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New branching rules induced by plethysm*

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Abstract

We derive group branching laws for formal characters of subgroups H_π of $\mathsf{GL}(n)$ leaving invariant an arbitrary tensor T^π of Young symmetry type π where π is an integer partition. The branchings $\mathsf{GL}(n) \downarrow \mathsf{GL}(n-1)$, $\mathsf{GL}(n) \downarrow \mathsf{O}(n)$ and $\mathsf{GL}(2n) \downarrow \mathsf{Sp}(2n)$ fixing a vector v_i, a symmetric tensor $g_{ij} = g_{ji}$ and an antisymmetric tensor $f_{ij} = -f_{ji}$, respectively, are obtained as special cases. All new branchings are governed by Schur function series obtained from plethysms of the Schur function $s_\pi \equiv \{\pi\}$ by the basic M series of complete symmetric functions and the $L = M^{-1}$ series of elementary symmetric functions. Our main technical tool is that of Hopf algebras and our main result is the derivation of a coproduct for any Schur function series obtained by plethysm from another such series. Therefrom one easily obtains π-generalized Newell–Littlewood formulae and the algebra of the formal group characters of these subgroups is established. Concrete examples and extensive tabulations are displayed for H_{1^3}, H_{21} and H_3, showing their involved and nontrivial representation theory. The nature of the subgroups is shown to be in general affine and in some instances non-reductive. We discuss the complexity of the coproduct formula and give a graphical notation to cope with it. We also discuss the way in which the group branching laws can be reinterpreted as twisted structures deformed by highly nontrivial 2-cocycles. The algebra of subgroup characters is identified as a cliffordization of the algebra of symmetric functions for $\mathsf{GL}(n)$ formal characters. Modification rules are beyond the scope of the present paper, but are briefly discussed.

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* Dedicated to the memory of our co-author, friend and colleague Brian Wybourne, 1935–2003.

This work is dedicated to the memory of our friend and colleague Brian G Wybourne, who was born on 3rd May 1935 in Morrinsville, New Zealand and died on 26th November 2003 in Torun, Poland. It was Brian's typically inquisitive response to a draft copy of the 'Hopf laboratory for symmetric functions' paper of BF and PDJ [11] that encouraged him together with RCK, in the autumn of 2003 while Brian was on one of his annual visits to Southampton, to look for examples of non-classical Lie groups with 'new' branching rules. That discussion led on to the present joint collaboration. We trust that our results, although only an initial foray into a technically difficult area of combinatorial representation theory, are in keeping with Brian's vision for some of the last research that occupied him before his untimely death.

Contents

1. Introduction

The use of explicit tensorial notation for handling the representation theory of the classical groups is a natural tool in many branches of mathematics and of mathematical or theoretical physics. The theory was formalized by Weyl [38] and the associated character techniques involving applications of symmetric functions developed especially by Littlewood [25]. For modern accounts of combinatorial representation theory, and its connections to other branches of mathematics, we refer to the review article of Barcelo and Ram [2] and for the theory of symmetric functions to the classic text of Macdonald [32]. A milestone in practical applications of these techniques is the survey paper [3] which gives systematic rules for handling partition notation for labelling (finite dimensional) representations of simple Lie groups, together with their branching rules to common subgroups, and for the resolution of their Kronecker products. An example of applications of Kronecker products is provided by [14]. A necessary concomitant of these techniques is the automation provided by a symbolic computer package such as SCHUR© [39]. Finally, some of these techniques have been found to generalize to the representation theory of non-compact groups [19–21]

In a recent paper [11], the role of symmetric functions in relation to group representation theory has been re-considered from the viewpoint of the underlying Hopf algebraic structure. This structure is in fact well known in the combinatorial literature [37, 36, 34]. In [11], the formalism of branching rules was aligned with certain endomorphisms on the algebra of symmetric functions, called branching operators, derived from 1-cochains, for which the multiplicative cohomology of Sweedler [35] provides a natural analytical setting and classification. Standard branchings from generic symmetric functions to symmetric functions of orthogonal or symplectic type (the classic Newell–Littlewood theorems for the group reduction from $\mathsf{GL}(n) \downarrow \mathsf{O}(n)$ or $\mathsf{GL}(n) \downarrow \mathsf{Sp}(n)$) were found to be derived from certain 2-cocycles (for which associativity is guaranteed).

This result then prompts the more general question of classifying arbitrary, noncohomologous, 2-cocycles, and the nature of any associated character theory and of the algebraic or group structures which might be entrained therewith. While an *ab initio* approach to this question is extremely difficult at this level, there is an obvious strategy for finding such generalizations. Namely one should look for 'branching rules' which are a direct generalization of those due to Littlewood and for which there is a known underlying classical matrix group. Apart from exceptional groups associated with specific invariants in sporadic dimensions, cases arising from local isomorphisms, and also the $\mathsf{GL}(n) \downarrow \mathsf{SL}(n)$ family (see section 3), such groups are necessarily not classical groups, as these are already exhausted by the orthogonal and symplectic series. Since one studies group characters associativity is guaranteed, and the associated 2-cochains are once again 2-cocycles.

This infinite reservoir of 'new branching rules' is revealed by simply looking for formal characters associated with matrix subgroups of $\mathsf{GL}(n)$ which fix a certain tensor T_π of arbitrary symmetry type, say corresponding to a partition π. The corresponding branching rules from $\mathsf{GL}(n)$ to such subgroups thus form the topic of investigation in the present paper.

The plan of the paper is as follows. In section 2, we introduce the main propositions about coproducts of infinite Schur function series, quotients of products by such series and products of formal characters of subgroups, all based on plethysms of the M-series. These results then form the basis of the calculation of 'new branching rules' from $\mathsf{GL}(n)$ to some matrix subgroups H_π that are exemplified in section 3, with both modification rules and product rules for H_{1^3} for dimension $n = 3, 4$. Further tabulations are provided for the cases H_{21} and H_3 in appendix C. All this is presented in the standard notation for symmetric functions, in a purely combinatorial way. Then we switch in section 4 to the much more handy and compact Hopf algebra language, giving a Hopf algebra proof of the same results. Thereafter we give only Hopf proofs in the main text and collect the combinatorial proofs in appendix B. Technically, the new branching rules are generated by skewing with certain infinite formal series of symmetric functions which are generalizations of those used by Littlewood, and some related ones needed for handling such cases as spinor and composite tensor representations. From the analysis we also derive the π-generalised Newell–Littlewood product formula (proposition 2.7, theorem 4.19).

An underlying theme of this work is the use of Littlewood's operation of plethysm [29, 25] of symmetric functions which emerges as pivotal to the generation of the new branching rules. Plethysms are defined in section 2, and in all subsequent sections play a central role. Specifically, in section 4.2 key results concerning plethysms of Schur function series are derived (main theorems (i) and (ii), theorems 4.13 and 4.14 respectively); in section 4.4 it is shown how plethysms allow one to construct an infinity of noncohomologous 2-cochains. A graphical calculus of tangle diagrams for interpreting this analysis is given in appendix A. A further study of plethysms from the viewpoint of the Hopf algebra structure of symmetric functions, as developed in [11], is in preparation [12].

2. Schur functions and Schur function series

2.1. Basic notions

As mentioned in the introduction, in this work we are dealing with formal group characters, which generically may be regarded as particular sorts of symmetric functions. Abstractly these lie in the ring of invariant polynomials $\Lambda^n = \mathbb{Z}[x_1, x_2, \ldots, x_n]^{S_n}$ in indeterminates $x_1, x_2, \ldots, x_n$ which are symmetric in their arguments. Via an inductive limit one passes over to infinitely many variables obtaining a graded ring $\Lambda = \oplus\Lambda^n$. As is well known, there are many bases for symmetric functions whose products and sums suffice to build up arbitrary symmetric polynomials [32]. Of particular interest for this paper are the so-called Schur functions. With an appropriate interpretation of $x = (x_1, x_2, \ldots, x_n)$, the Schur function $s_\lambda(x)$, or $\{\lambda\}$ in the notation of Littlewood [25], is the character of the irreducible representation V^λ of $\mathsf{GL}(n)$ of highest weight λ. In this notation λ is an integer partition, that is $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, an ordered ℓ-tuple of positive integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_\ell$; λ is a partition of $\omega_\lambda \equiv |\lambda| := \sum_i \lambda_i$ and the number of parts or *length* of the partition is $\ell(\lambda)$ or just ℓ in this case.

The ring of symmetric functions $\Lambda = \mathbb{Z}[x_1, x_2, \ldots]$ admits various operations, such as addition and (outer) multiplication. The latter gives rise in the Schur function basis to the famous Littlewood–Richardson coefficients as the structure constants in the product

$$s_\lambda(x) \cdot s_\mu(x) = \sum_\nu C_{\lambda\mu}{}^\nu s_\nu(x). \tag{2.1}$$

Reciprocally, there is an adjoint (outer) skew operation such that

$$s_{\nu/\mu}(x) = \sum_\lambda C_{\lambda\mu}{}^\nu s_\lambda(x). \tag{2.2}$$

Additional symmetric function definitions will be introduced in the following as they arise. It should be noted that we are flexible as to whether the discussion covers *universal characters*, where the number of indeterminates is formally infinite. Then the polynomial ring is being extended to the closure using the inductive limit of letting the number of variable tend to infinity. Doing that, one knows that so-called syzygies are avoided, and such formal representation modules are flat. By contrast for symmetric functions with a finite number n of variables, for example model subgroups of $\mathsf{GL}(n)$, one hits the problem of dealing with such syzygies. This amounts to introducing further relations, the so-called modification rules. A systematic treatment of modification rules for non-classical groups is beyond the scope of the paper, but we will exhibit a few instances to show how to cope with them in practice. Generally speaking our aim is to display a variety of examples for specific matrix groups, for which a finite number of variables is necessary and intended. However, one should keep in mind that abstract arguments, for example in section 4, are dealt with in the infinite variable case. This is common practice in the classical theory too.

2.2. Plethysm

One more advanced piece of symmetric function formalism, that of plethysm, is central to the structure of Schur function series and branching rules old and new. We give notation and some formal definitions here. We denote Schur functions as s_λ or $\{\lambda\}$ in Littlewood's bracket notation.

2.2.1. Plethysm as composition. The mathematical definition of plethysm is given by the composition of Schur functions, $s_\mu[s_\lambda]$. Using Littlewood's notation for Schur functions $s_\lambda \equiv \{\lambda\}$, it is customary to write a plethysm using the symbol for a tensor product $s_\mu[s_\lambda] \equiv \{\lambda\} \underline{\otimes} \{\mu\}$ with reversed terms. We use the underlined tensor product symbol $\underline{\otimes}$ for plethysm, to make a distinction with the tensor sign $\otimes$ appearing in the Hopf algebra development below. We do not use the $\circ$ as symbol for composition either, since this symbol is frequently employed for inner products, which we however denote by $\star$. The 'composition' definition of plethysm may be explained as follows in the context of the formal combinatorial definition of a Schur function [32]. A Schur function is given by

$$s_\lambda(x) = \sum_{T \in ST^\lambda} x^{\mathrm{wgt}(T)}, \tag{2.3}$$

where the sum is over all tableaux (fillings) T belonging to the set ST^λ of semi-standard tableaux of shape λ. Each summand is a monomial in the variables $x_1, x_2, \ldots, x_n$. If there are m such monomials in the Schur function $s_\lambda(x)$, and these are denoted by y_i with $i = 1, 2, \ldots, m$, then the plethysm, or composition, of the Schur function s_μ with the Schur function s_λ, is given by

$$s_\mu[s_\lambda](x) = s_\mu(y) = \sum_{T \in ST^\mu} y^{\mathrm{wgt}(T)} \tag{2.4}$$

where the entries in each tableau are now taken from the set $\{y_i \mid i = 1, 2, \ldots, m\}$ of monomials of the Schur function $s_\lambda(x)$.

Example 2.1. Consider $s_{(2)}[s_{(1^2)}](x_1, \ldots, x_4)$. Expand $s_{(1^2)}$ as

$$\begin{aligned} s_{(1^2)}(x_1, \ldots, x_4) &= x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \\ &= y_1 + y_2 + y_3 + y_4 + y_5 + y_6 \end{aligned} \tag{2.5}$$

which leads to the expansion of the composition $s_{(2)}[s_{(1^2)}]$ as

$$\begin{aligned} s_{(2)}[s_{(1^2)}](x_1, \ldots, x_4) &= s_{(2)}(y_1, \ldots, y_6) = s_{(2)}(x_1x_2, \ldots, x_3x_4) \\ &= y_1^2 + \cdots + y_6^2 + y_1y_2 + \cdots + y_5y_6 \\ &= x_1^2x_2^2 + \cdots + x_3^3x_4^2 + x_1^2x_2x_3 + \cdots + x_4^2x_2x_3 + 3x_1x_2x_3x_4 \\ &= s_{(2^2)}(x_1, \ldots, x_4) + s_{(1^4)}(x_1, \ldots, x_4). \end{aligned} \tag{2.6}$$

Here, the problem of the evaluation of the plethysm is simply to expand $s_{(2)}(y)$ in the Schur function basis $s_\nu(x)$ with ν a partition of 4.

2.2.2. Plethysms related to branchings. Consider two groups $\mathsf{GL}(m)$, $\mathsf{GL}(n)$, with $m > n$. Consider a Schur function $\{\lambda\}$ which represents the character of an irreducible m-dimensional representation of $\mathsf{GL}(n)$. This representation can be surjectively embedded in the fundamental representation of the group $\mathsf{GL}(m)$ whose character is $\{1\}$. The branching process $\mathsf{GL}(m) \to \mathsf{GL}(n)$ is then described by the injective map $\{1\} \to \{\lambda\}$ which leads to the general formula

$$\mathsf{GL}(m) \to \mathsf{GL}(n): \quad \{\mu\} \to \{\lambda\} \underline{\otimes} \{\mu\}. \tag{2.7}$$

This process, intimately related to physics, is the origin of the usage of the tensor symbol $\underline{\otimes}$. The connection with the previous definition of a plethysm as a composition comes about because the $\mathsf{GL}(n)$ character $\{\lambda\}$ is nothing other than the Schur function $s_\lambda(x)$, with a suitable identification of x. Its dimension m, obtained by setting all $x_i = 1$, is just the number of monomials y_i in $s_\lambda(x)$, and $s_\mu(y)$ is the corresponding $\mathsf{GL}(m)$ character $\{\mu\}$.

In terms of modules, let V^λ be the $\mathsf{GL}(n)$-module with character $\{\lambda\}$ and dimension m. This module may be identified with the defining $\mathsf{GL}(m)$-module V on which $\mathsf{GL}(m)$ acts naturally. Then the plethysm $\{\lambda\} \underline{\otimes} \{\mu\}$ arises as the character of the $\mathsf{GL}(m)$-module $V^\mu = (V^\lambda)^\mu$ viewed as a $\mathsf{GL}(n)$-module. As a result of this interpretation, it is sometimes convenient to adopt a notation for plethsyms whereby the corresponding character is denoted not by $\{\lambda\} \underline{\otimes} \{\mu\}$, but by $\{\lambda\}^{\underline{\otimes}\{\mu\}}$.

2.2.3. Plethysms and outer exponentiation. Plethysm can be tied to the following problem of invariants of matrices $|A^\lambda|$. For example, the characteristic polynomial gives a relationship between the roots x_i, that is invariants, and the coefficients of the polynomials. The process of evaluating a plethysm is the same as that of computing the coefficients of the characteristic polynomial having roots x_i^k from the coefficients of the original polynomial. In general, one tries to compute the invariant $|A^\lambda|^\mu$ where μ is a partition of k. The relation to outer product tensor powers is then as follows. The plethysm

$$\{\lambda\} \underline{\otimes} \{\mu\} = \sum_\nu p^\nu_{\lambda,\mu} \{\nu\}, \tag{2.8}$$

with non-negative integers $p^\nu_{\lambda,\mu}$, appears in the outer tensor product k-fold power of the Schur function $\{\lambda\}$ given by

$$\{\lambda\}^k = \sum_{\mu, |\mu|=k} f^\mu \{\lambda\} \underline{\otimes} \{\mu\}, \tag{2.9}$$

where the multiplicity f^μ is the dimension of the irreducible representation $\{\mu\}$ of the symmetric group S_k. This makes it clear that the plethysm $\{\lambda\} \underline{\otimes} \{\mu\}$ is nothing other than the character of the $\{\mu\}$-symmetrized tensor power of the $\mathsf{GL}(n)$-module V^λ, justifying yet again the exponential notation $\{\lambda\}^{\underline{\otimes}\{\mu\}}$ that we will meet in our main theorem. As iterated outer multiplication, that is exponentiation, the operation of plethysm is of course not commutative, and satisfies various forms of right and left distributivities, which in fact can be used to compute plethysms iteratively. Details are deferred to section 4.2, where the results are needed.

2.3. Schur function series

Littlewood [25] introduced a set of infinite Schur function series which much later allowed King [17] to formulate various identities and branching rules in an extremely compact notation. These identities have been extended by King, Dehuai and Wybourne [18] and later by Yang and Wybourne [40]; we follow the presentation of the latter[5].

A Schur function series is an infinite series of Schur functions often given via a generating function defining a formal power series. The most basic Schur function series are the mutually inverse series L and $M = L^{-1}$.

$$L_t(x) = \prod_{i=1}^{\infty}(1 - x_i t) \qquad M_t(x) = \prod_{i=1}^{\infty}(1 - x_i t)^{-1} = L_t^{-1}(x) \tag{2.10}$$

[5] There are misprints in equations (2), (3) of [40] and a weak notation which unfortunately was kept in [11]. To cope notationally with the transformations ˜, †, $^{-1}$, one needs to consider the formal parameter t of the series.

Table 1. Schur function series: Type, Name = Product, Schur function content and plethysm.

L	$L_t = \prod_i (1 - x_i t)$	$\sum_m (-1)^m \{1^m\} t^m$	$L_t(x_i)$	$\{1\} \underline{\otimes} L_t$
L^{-1}	$M_t = \prod_i (1 - x_i t)^{-1}$	$\sum_m \{m\} t^m$	$L_t(x_i)^{-1}$	$(-\{1\}) \underline{\otimes} L_t$
$L^\dagger$	$Q_t = \prod_i (1 + x_i t)$	$\sum_m \{1^m\} t^m$	$L_{-t}(x_i)$	$\{1\} \underline{\otimes} L_{-t}$
$\tilde{L}$	$P_t = \prod_i (1 + x_i t)^{-1}$	$\sum_m (-1)^m \{m\} t^m$	$L_{-t}(x_i)^{-1}$	$(-\{1\}) \underline{\otimes} L_{-t}$
A	$A_t = \prod_{i<j} (1 - x_i x_j t)$	$\sum_\alpha (-1)^{\omega_\alpha/2} \{\alpha\} t^{\omega_\alpha}$	$L_t(x_i x_j)(i < j)$	$\{1^2\} \underline{\otimes} L_t$
A^{-1}	$B_t = \prod_{i<j} (1 - x_i x_j t)^{-1}$	$\sum_\beta \{\beta\} t^{\omega_\beta}$	$L_t(x_i x_j)^{-1}(i < j)$	$(-\{1^2\}) \underline{\otimes} L_t$
$\tilde{A}$	$C_t = \prod_{i\leqslant j} (1 - x_i x_j t)$	$\sum_\gamma (-1)^{\omega_\gamma/2} \{\gamma\} t^{\omega_\gamma}$	$L_t(x_i x_j)(i \leqslant j)$	$\{2\} \underline{\otimes} L_t$
$A^\dagger$	$D_t = \prod_{i\leqslant j} (1 - x_i x_j t)^{-1}$	$\sum_\delta \{\delta\} t^{\omega_\delta}$	$L_t(x_i x_j)^{-1}(i \leqslant j)$	$(-\{2\}) \underline{\otimes} L_t$
$V = \tilde{V}$	$V_t = \prod_i (1 - x_i^2 t)$	$\sum_{p,q} (-1)^p \{\widetilde{p + 2q, p}\} t^{p+q}$	$L_t(x_i^2)$	$(\{2\} - \{1^2\}) \underline{\otimes} L_t$
$V^{-1} = V^\dagger$	$W_t = \prod_i (1 - x_i^2 t)^{-1}$	$\sum_{p,q} (-1)^p \{p + 2q, p\} t^{p+q}$	$L_t(x_i^2)^{-1}$	$(\{1^2\} - \{2\}) \underline{\otimes} L_t$

from which the others may be derived using plethysm, sum and product. The Schur function content of these mutually inverse series is

$$
\begin{aligned}
L_t(x) &= \sum_{m=0}^{\infty} (-1)^m s_{(1^m)}(x) t^m = \sum_{m=0}^{\infty} (-1)^m \{1^m\} t^m, \\
M_t(x) &= \sum_{m=0}^{\infty} s_{(m)}(x) t^m = \sum_{m=0}^{\infty} \{m\} t^m,
\end{aligned}
\tag{2.11}
$$

again using the notation $\{\lambda\}$ of Littlewood for a Schur function $s_\lambda(x)$.

Furthermore, it is convenient to follow Yang and Wybourne to introduce the conjugate (with respect to transposed partitions) series, signified by $\tilde{}$, and the inverse conjugate or adjoint series, signified by $\dagger$:

$$
\begin{aligned}
&P_t(x) = \tilde{L}_t(x) = L_{-t}(x)^{-1} = \prod_{i=1}^{\infty} (1 + x_i t)^{-1} = \sum_{m=0}^{\infty} (-1)^m \{m\} t^m, \\
&Q_t(x) = L_t^\dagger(x) = (\tilde{L}_t(x))^{-1} = \widetilde{L_t^{-1}}(x) = L_{-t}(x) = \prod_{i=1}^{\infty} (1 + x_i t) = \sum_{m=0}^{\infty} \{1^m\} t^m.
\end{aligned}
\tag{2.12}
$$

Note that taking the adjoint is equivalent to the transformation $t \to -t$, while the inversion $L \to L^{-1}$ can be viewed as a plethysm

$$
L_t^{-1}(x) = (-\{1\}) \underline{\otimes} L_t \qquad \tilde{L}_t(x) = (-\{1\}) \underline{\otimes} L_{-t}. \tag{2.13}
$$

Having the counit available (see below) one can recast this into the form $(-\{\nu\}) \underline{\otimes} \{\mu\} = \{\nu\} \underline{\otimes} \mathsf{S}(\{\mu\}) = \{\nu\} \underline{\otimes} [(-1)^{|\mu|} \{\mu'\}]$ with $\{\mu'\}$ the conjugate partition. The other series[6] and their forms as plethysms are then derived in a similar manner, see [40]. We may be lax about the index t since we are dealing mainly with the series L and M. Generally, Schur function series come in pairs which are mutually inverse and consecutively named. One finds

$$
AB = CD = EF = GH = LM = PQ = RS = VW = 1. \tag{2.14}
$$

Generating functions and relations between some of these are displayed in table 1, following [40]. While we are following in this section the habit found in the literature to derive Schur function series from the L series, the forthcoming sections are based on the M series to avoid frequent usage of the clumsy notation L^{-1}.

[6] The series can be considered to belong to the ring of formal power series $\Lambda[[t]]$ associated to the ring of symmetric functions Λ, which admits a λ-ring structure [32, 22].

Some remaining series are $E = LA$, $F = L^{-1}A^{-1}$, $G = L^{\dagger}A$, $H = \tilde{L}A^{-1}$, $R = L\tilde{L}$, $S = L^{-1}L^{\dagger}$, $V = \tilde{A}A^{-1}$ and $W = AA^{\dagger}$. In the table, the following are the characterising properties of the partitions that have been used: (α) is given in Frobenius notation by

$$(\alpha) = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ a_1+1 & a_2+1 & \dots & a_r+1 \end{pmatrix} \tag{2.15}$$

and $\{\gamma\}$ is its conjugate (obtained in Frobenius notation by interchanging the two rows); $\{\delta\}$ has only even parts and $\{\beta\}$ is its conjugate; $\{\epsilon\}$ (not in the table, but related to E) is self-conjugate; $\{\zeta\}$ (not in the table, but related to F) is an arbitrary partition.

We will use $(\!(\cdots)\!)_\pi$ as character brackets for the infinitely many new types of characters derived using $M_\pi = \{\pi\} \underline{\otimes} M$ series in the following. Often, the index π is replaced by the dimension of the representation $(\!(\cdots)\!)_{\mathrm{dim}}$, but confusion should not occur.

The branching rules for the restriction from $\mathsf{GL}(n)$ to the subgroups $\mathsf{GL}(n-1)$, $\mathsf{O}(n)$ and $\mathsf{Sp}(n)$ (for n even) are given by skewing the Schur functions corresponding to characters of $\mathsf{GL}(n)$ by various infinite Schur function series,

$$\mathsf{GL}(n) \supset \mathsf{GL}(n-1) \qquad \{\lambda\} \to \{\lambda/(\{1\}\underline{\otimes} M)\} = \{\lambda/M\} = (\!(\lambda/M_1)\!)_{\{1\}}, \tag{2.16}$$

$$\mathsf{GL}(n) \supset \mathsf{O}(n) \qquad \{\lambda\} \to [\lambda/(\{2\}\underline{\otimes} M)] = [\lambda/D] = (\!(\lambda/M_2)\!)_{\{2\}}, \tag{2.17}$$

$$\mathsf{GL}(n) \supset \mathsf{Sp}(n) \qquad \{\lambda\} \to \langle\lambda/(\{1^2\}\underline{\otimes} M)\rangle = \langle\lambda/B\rangle = (\!(\lambda/M_{1^2})\!)_{\{1^2\}}. \tag{2.18}$$

The origin of these rules is the fact that $\mathsf{GL}(n-1)$, $\mathsf{O}(n)$ and $\mathsf{Sp}(n)$ are the subgroups of $\mathsf{GL}(n)$ that leave invariant a vector, a symmetric second rank tensor and an antisymmetric second rank tensor, or in a basis for $\mathbb{C}^n$, objects v_i, $g_{ij} = g_{ji}$, $f_{ij} = -f_{ji}$, respectively. It is natural to ask what are the subgroups of $\mathsf{GL}(n)$ that leave invariant higher rank tensors, such as a third rank fully antisymmetric tensor[7] $\eta_{ijk} = -\eta_{jik} = -\eta_{ikj}$ would have Young type $\{1^3\}$. If we denote the corresponding subgroup[8] by $\mathsf{H}_{1^3}(n)$, then the corresponding branching rule is formally given by

$$\mathsf{GL}(n) \supset \mathsf{H}_{1^3}(n) \qquad \{\lambda\} \to (\!(\lambda/(\{1^3\}\underline{\otimes} M))\!) = (\!(\lambda/M_{1^3})\!)_{1^3}. \tag{2.19}$$

More generally, if $\mathsf{H}_\pi(n)$ is the subgroup of $\mathsf{GL}(n)$ leaving invariant a tensor whose symmetry is specified by the partition π, we have[9]

$$\mathsf{GL}(n) \supset \mathsf{H}_\pi(n) \qquad \{\lambda\} \to (\!(\lambda/(\{\pi\}\underline{\otimes} M))\!)_\pi = (\!(\lambda/M_\pi)\!)_\pi. \tag{2.20}$$

These formal identities may be inverted through the use of $L = M^{-1}$ to give

$$\mathsf{GL}(n-1) \subset \mathsf{GL}(n) \qquad \{\lambda\} \to \{\lambda/(\{1\}\underline{\otimes} L)\} = \{\lambda/L\} = \left\{\lambda/M_1^{-1}\right\}, \tag{2.21}$$

$$\mathsf{O}(n) \subset \mathsf{GL}(n) \qquad [\lambda] \to \{\lambda/(\{2\}\underline{\otimes} L)\} = \{\lambda/C\} = \left\{\lambda/M_2^{-1}\right\}, \tag{2.22}$$

$$\mathsf{Sp}(n) \subset \mathsf{GL}(n) \qquad \langle\lambda\rangle \to \{\lambda/(\{1^2\}\underline{\otimes} L)\} = \{\lambda/A\} = \left\{\lambda/M_{1^2}^{-1}\right\}, \tag{2.23}$$

$$\mathsf{H}_{1^3}(n) \subset \mathsf{GL}(n) \qquad (\!(\lambda)\!) \to \{\lambda/(\{1^3\}\underline{\otimes} L)\} = \left\{\lambda/M_{1^3}^{-1}\right\}, \tag{2.24}$$

and, more generally, for the subgroup of type $\mathsf{H}_\pi(n)$,

$$\mathsf{H}_\pi(n) \subset \mathsf{GL}(n) \qquad (\!(\lambda)\!)_\pi \to \{\lambda/(\{\pi\}\underline{\otimes} L)\} = \{\lambda/L_\pi\} = \left\{\lambda/M_\pi^{-1}\right\}. \tag{2.25}$$

[7] There seem to be only a few instances of research on this or related topics, for example [9].

[8] In principle we can consider subgroups which leave invariant a linear combination of tensors of different Young symmetry types, e.g. $\{1^2\}\{1\} = \{1^3\} + \{21\}$, but we postpone to investigate this complication until section 4.

[9] Analogously to branching by skewing, formal multiplicative branchings, for example $\{\lambda\} \mapsto (\!(\lambda \cdot M)\!)$, $\{\lambda\} \mapsto (\!(\lambda \cdot D)\!)$, $\{\lambda\} \mapsto (\!(\lambda \cdot B)\!)$, arise in the case of the branching of unitary representations of non-compact Lie groups [19, 20, 40] to representations of a maximal compact Lie subgroup. An analogous generalization of multiplicative branching, for example $\{\lambda\} \mapsto (\!(\lambda \cdot \{\pi\}\underline{\otimes} M)\!)$, might equally be considered.

In the following subsections, we give some of the systematics of general series Φ, including M_π and their inverses M_π^{-1} as special cases. These series encode the complexity necessary to deal with the manipulation of formal characters $\{\lambda\}_\Phi$, including as a special case for the M_π series, the formal characters $(\!(\lambda)\!)_\pi$. Having done this, we will be in a position to examine specific cases of non-classical subgroups H_π as an illustration of the general theme of the paper. In the course of this work, we pause to recall a few notions about the outer Hopf algebra structure of symmetric functions. Switching to a more advanced description will ease the presentation and proofs; appendix B provides the combinatorial details for the more conservative reader.

2.4. 'Coproducts' of Schur function series: combinatorics

As will be seen below, a crucial part of the manipulations with branching rules is associated with re-writing a symmetric function of a set of indeterminates $z = (z_1, z_2, \ldots)$ in terms of symmetric functions of its parts, if it is regarded as partitioned into two subsets $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$, or in the finite-dimensional case $z = (x, y) = (x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n)$. For reasons to be explained below, this expansion is here called a 'coproduct'. For the moment, we simply proceed with the explicit steps.

Proposition 2.2. *It is well known that*

$$M(x, y) = M(x)M(y), \tag{2.26}$$

$$L(x, y) = L(x)L(y), \tag{2.27}$$

$$A(x, y) = A(x)A(y)\sum_\sigma(-1)^{|\sigma|}s_\sigma(x)s_{\sigma'}(y), \tag{2.28}$$

$$B(x, y) = B(x)B(y)\sum_\sigma s_\sigma(x)s_\sigma(y), \tag{2.29}$$

$$C(x, y) = C(x)C(y)\sum_\sigma(-1)^{|\sigma|}s_\sigma(x)s_{\sigma'}(y), \tag{2.30}$$

$$D(x, y) = D(x)D(y)\sum_\sigma s_\sigma(x)s_\sigma(y), \tag{2.31}$$

where $|\sigma| \equiv \omega_\sigma$ is the weight of the partition σ and σ' denotes the conjugate of σ.

Proof. The first of these results (2.26) corresponds to the trivial observation that

$$M(x, y) = \prod_{i=1}^{m}(1 - x_i)^{-1}\prod_{a=1}^{n}(1 - y_a)^{-1} = M(x)M(y). \tag{2.32}$$

A similar observation immediately gives (2.27). The derivations of (2.28)–(2.31) depend on the Cauchy identity and its inverse:

$$\mathsf{C}(x, y) = \prod_{\substack{1\leqslant i\leqslant m\\1\leqslant a\leqslant n}}(1 - x_i y_a)^{-1} = \sum_\sigma s_\sigma(x)s_\sigma(y), \tag{2.33}$$

$$\mathsf{C}^{-1}(x, y) = \prod_{\substack{1\leqslant i\leqslant m\\1\leqslant a\leqslant n}}(1 - x_i y_a) = \sum_\sigma(-1)^{|\sigma|}s_\sigma(x)s_{\sigma'}(y). \tag{2.34}$$

These can be used to give, for example,

$$B(x,y) = \prod_{1\leqslant i<j\leqslant m}(1-x_ix_j)^{-1}\prod_{\substack{1\leqslant i\leqslant m\\1\leqslant a\leqslant n}}(1-x_iy_a)^{-1}\prod_{1\leqslant a<b\leqslant n}(1-y_ay_b)^{-1}$$
$$= B(x)\left(\sum_\sigma s_\sigma(x)s_\sigma(y)\right)B(y), \tag{2.35}$$

thereby proving (2.29). The remaining results follow in the same way. □

Similarly, using (2.33) more than once in the case of $M_{1^3}(x)$ we find

$$M_{1^3}(x,y) = M_{1^3}(x)M_{1^3}(y)\sum_{\sigma,\tau}s_\sigma(x)s_{\{1^2\}\underline{\otimes}\{\tau\}}(x)s_{\{1^2\}\underline{\otimes}\{\sigma\}}(y)s_\tau(y). \tag{2.36}$$

More generally, we can formulate the important

Proposition 2.3. *For any partition* π

$$M_\pi(x,y) = M_\pi(x)M_\pi(y)\prod_{\xi,\eta<\pi}\prod_{k=1}^{C^\pi_{\xi\eta}}\sum_{\sigma(\xi,\eta,k)}s_{\xi\underline{\otimes}\sigma(\xi,\eta,k)}(x)s_{\eta\underline{\otimes}\sigma(\xi,\eta,k)}(y), \tag{2.37}$$

where the coefficients $C^\pi_{\xi\eta}$ *are the Littlewood–Richardson coefficients defined by*

$$s_\xi(x)s_\eta(x) = \sum_\pi C^\pi_{\xi\eta}s_\pi(x), \tag{2.38}$$

or, equivalently,

$$s_\pi(x,y) = \sum_{\xi,\eta}C^\pi_{\xi\eta}s_\xi(x)s_\eta(y). \tag{2.39}$$

Proof. Let $N^\pi = \sum_{\xi\eta}C^\pi_{\xi\eta}$ be the number of summands in the expansion of $s_\pi(x,y)$ in the form (2.39). This includes the two summands $s_\pi(x)$ and $s_\pi(y)$, which may conveniently be taken to be the first and the last, respectively, corresponding to the cases $(\xi,\eta) = (\pi,0)$ and $(\xi,\eta) = (0,\pi)$, which occur with multiplicity 1. Then,

$$s_\pi(x,y) = \sum_{\xi,\eta}C^\pi_{\xi\eta}s_\xi(x)s_\eta(y) = \sum_{k=1}^{N^\pi}s_{\xi(k)}(x)s_{\eta(k)}(y). \tag{2.40}$$

It follows that

$$M_\pi(x,y) = \prod_{T\in ST^\pi}(1-(x,y)^{\mathrm{wgt}(T)})^{-1}$$
$$= \prod_{k=1}^{N^\pi}\prod_{U\in ST^{\xi(k)}}\prod_{V\in ST^{\eta(k)}}(1-x^{\mathrm{wgt}(U)}y^{\mathrm{wgt}(V)})^{-1}$$
$$= \prod_{k=1}^{N^\pi}\sum_{\sigma(k)}s_{\sigma(k)}[s_{\xi(k)}](x)s_{\sigma(k)}[s_{\eta(k)}](y)$$
$$= M_\pi(x)M_\pi(y)\prod_{k=2}^{N^\pi-1}\sum_{\sigma(k)}s_{\xi(k)\underline{\otimes}\sigma(k)}(x)s_{\eta(k)\underline{\otimes}\sigma(k)}(y)$$
$$= M_\pi(x)M_\pi(y)\prod_{\xi,\eta<\pi}\prod_{k=1}^{C^\pi_{\xi\eta}}\sum_{\sigma(\xi,\eta,k)}s_{\xi\underline{\otimes}\sigma(\xi,\eta,k)}(x)s_{\eta\underline{\otimes}\sigma(\xi,\eta,k)}(y), \tag{2.41}$$

where in the second step use has been made of Cauchy's identity (2.33), while in the third it has been assumed that $k = 1$ and $k = N^\pi$ correspond to the two summands $s_\pi(x)$ and $s_\pi(y)$, respectively. □

Example 2.4. It follows from proposition 2.3 that

$$M_3(x, y) = M_3(x)M_3(y)\sum_{\sigma,\tau} s_\sigma(x)s_{\{2\}\underline{\otimes}\tau}(x)\cdot s_{\{2\}\underline{\otimes}\sigma}(y)s_\tau(y), \tag{2.42}$$

$$M_{21}(x, y) = M_{21}(x)M_{21}(y)\sum_{\alpha,\beta,\gamma,\delta} s_\alpha(x)s_\beta(x)s_{\{1^2\}\underline{\otimes}\gamma}(x)s_{\{2\}\underline{\otimes}\delta}(x) \cdot s_{\{1^2\}\underline{\otimes}\alpha}(y)s_{\{2\}\underline{\otimes}\beta}(y)s_\gamma(y)s_\delta(y), \tag{2.43}$$

$$M_{1^3}(x, y) = M_{1^3}(x)M_{1^3}(y)\sum_{\sigma,\tau} s_\sigma(x)s_{\{1^2\}\underline{\otimes}\tau}(x)\cdot s_{\{1^2\}\underline{\otimes}\sigma}(y)s_\tau(y), \tag{2.44}$$

$$M_{1^4}(x, y) = M_{1^4}(x)M_{1^4}(y)\sum_{\rho,\sigma,\tau} s_\rho(x)s_{\{1^2\}\underline{\otimes}\sigma}(x)s_{\{1^3\}\underline{\otimes}\tau}(x)\cdot s_{\{1^3\}\underline{\otimes}\rho}(y)s_{\{1^2\}\underline{\otimes}\sigma}(y)s_\tau(y) \tag{2.45}$$

2.5. Series quotients of Schur function products

The previous results allow us to prove the following:

Proposition 2.5. *For any partitions π, μ and ν*

$$(\{\mu\}\{\nu\})/M_\pi = \sum_{\sigma(\xi,\eta,k)} \left\{\mu \Big/ \left(M_\pi \prod_{\xi,\eta<\pi}\prod_{k=1}^{C^\pi_{\xi\eta}} \xi\,\underline{\otimes}\,\sigma(\xi,\eta,k)\right)\right\} \times \left\{\nu \Big/ \left(M_\pi \prod_{\xi,\eta<\pi}\prod_{k=1}^{C^\pi_{\xi\eta}} \eta\,\underline{\otimes}\,\sigma(\xi,\eta,k)\right)\right\}, \tag{2.46}$$

where for each ξ, η and k the summation is carried out over all partitions $\sigma(\xi, \eta, k)$.

This proposition will be a corollary to our main theorem, so we postpone the proof. The reader who feels uneasy with our Hopf algebraic proof might like to compare the proof of proposition 2.5 by combinatorial means in appendix B and is invited to compare with the further development using Hopf algebras. It should be noted in particular that the lemmas B.20 and B.21 becoming implicit in the structural definitions of the Hopf algebraic machinery rather than requiring separate proofs.

Example 2.6. Proposition 2.5 encompasses by way of example the following results:

$$(\{\mu\}\{\nu\})/M = \{\mu/M\}\{\nu/M\}, \tag{2.47}$$

$$(\{\mu\}\{\nu\})/M_2 = \sum_\sigma \{\mu/(M_2\sigma)\}\{\nu/(M_2\sigma)\}, \tag{2.48}$$

$$(\{\mu\}\{\nu\})/M_{1^2} = \sum_\sigma \{\mu/(M_{1^2}\sigma)\}\{\nu/(M_{1^2}\sigma)\}, \tag{2.49}$$

$$(\{\mu\}\{\nu\})/M_3 = \sum_{\sigma,\tau} \{\mu/(M_3\sigma(2\,\underline{\otimes}\,\tau))\}\{\nu/(M_3(2\,\underline{\otimes}\,\sigma)\tau)\}, \tag{2.50}$$

$$(\{\mu\}\{\nu\})/M_{21} = \sum_{\alpha,\beta,\gamma,\delta} \{\mu/(M_{21}\alpha\beta(1^2 \underline{\otimes} \gamma)(2 \underline{\otimes} \delta))\}\{\nu/(M_{21}(1^2 \underline{\otimes} \alpha)(2 \underline{\otimes} \beta)\gamma\delta)\}, \tag{2.51}$$

$$(\{\mu\}\{\nu\})/M_{1^3} = \sum_{\sigma,\tau} \{\mu/(M_{1^3}\sigma(1^2 \underline{\otimes} \tau))\}\{\nu/(M_{1^3}(1^2 \underline{\otimes} \sigma)\tau)\}, \tag{2.52}$$

$$(\{\mu\}\{\nu\})/M_{1^4} = \sum_{\rho,\sigma,\tau} \{\mu/(M_{1^4}\rho(1^2 \underline{\otimes} \sigma)(1^3 \underline{\otimes} \tau))\}\{\nu/(M_{1^4}(1^3 \underline{\otimes} \rho)(1^2 \underline{\otimes} \sigma)\tau)\}. \tag{2.53}$$

2.6. *π-Newell–Littlewood product theorem*

Having identified formal characters of representations of subgroups $\mathsf{H}_\pi(n)$ of $\mathsf{GL}(n)$ in (2.25), we are in a position to combinatorially decompose their tensor products. This may be done by writing such products as products of Schur functions using (2.25), evaluating the Schur function products by means of (2.38) and then restricting the corresponding characters of $\mathsf{GL}(n)$ to its subgroup $\mathsf{H}_\pi(n)$ by means of (2.20). Our previous propositions allow us to simplify the results and obtain the general formula given in

Proposition 2.7. *Let* $(\!(\mu)\!)$ *and* $(\!(\nu)\!)$ *be formal characters of* H_π*. Then,*

$$(\!(\mu)\!)(\!(\nu)\!) = \sum_{\sigma(\xi,\eta,k)} \left(\!\!\left(\left\{ \mu \Big/ \prod_{\xi,\eta<\pi} \prod_{k=1}^{C^\pi_{\xi\eta}} \{\xi\} \underline{\otimes} \sigma(\xi,\eta,k) \right\} \left\{ \nu \Big/ \prod_{\xi,\eta<\pi} \prod_{k=1}^{C^\pi_{\xi\eta}} \{\eta\} \underline{\otimes} \sigma(\xi,\eta,k) \right\} \right)\!\!\right), \tag{2.54}$$

where for each ξ*,* η *and* k *the summation is carried out over all partitions* $\sigma(\xi,\eta,k)$*.*

The combinatorial proof is given in appendix B. The theorem will be an easy consequence of our main theorem in section 4.

Example 2.8. Once again proposition 2.7 may be illustrated by means of examples:

$$\mathsf{H}_1(n) = \mathsf{GL}(n-1): \quad \{\mu\}\{\nu\} = \{(\mu)(\nu)\}, \tag{2.55}$$

$$\mathsf{H}_2(n) = \mathsf{O}(n): \quad [\mu][\nu] = \sum_\sigma [(\mu/\sigma)(\nu/\sigma)], \tag{2.56}$$

$$\mathsf{H}_{1^2}(n) = \mathsf{Sp}(n): \quad \langle\mu\rangle\langle\nu\rangle = \sum_\sigma \langle(\mu/\sigma)(\nu/\sigma)\rangle, \tag{2.57}$$

$$\mathsf{H}_3(n): \quad (\!(\mu)\!)(\!(\nu)\!) = \sum_{\sigma,\tau} (\!((\mu/\sigma(2 \underline{\otimes} \tau))(\nu/(2 \underline{\otimes} \sigma)\tau))\!), \tag{2.58}$$

$$\mathsf{H}_{21}(n): \quad (\!(\mu)\!)(\!(\nu)\!) = \sum_{\alpha,\beta,\gamma,\delta} (\!((\mu/\alpha\beta(1^2 \underline{\otimes} \gamma)(2 \underline{\otimes} \delta)) \cdot (\nu/(1^2 \underline{\otimes} \alpha)(2 \underline{\otimes} \beta)\gamma\delta))\!), \tag{2.59}$$

$$\mathsf{H}_{1^3}(n): \quad (\!(\mu)\!)(\!(\nu)\!) = \sum_{\sigma,\tau} (\!((\mu/\sigma(1^2 \underline{\otimes} \tau))(\nu/(1^2 \underline{\otimes} \sigma)\tau))\!), \tag{2.60}$$

$$\mathsf{H}_{1^4}(n): \quad (\!(\mu)\!)(\!(\nu)\!) = \sum_{\rho,\sigma,\tau} (\!((\mu/\rho(1^2 \underline{\otimes} \sigma)(1^3 \underline{\otimes} \tau)) \cdot (\nu/(1^3 \underline{\otimes} \rho)(1^2 \underline{\otimes} \sigma)\tau))\!). \tag{2.61}$$

3. Nature of the non-classical groups H_π

Before developing the technicalities of our machinery in section 4, we try in the present section to give some hints as to what kinds of groups H_π are to be expected. The subgroups $\mathsf{H}_\pi(n)$ of $\mathsf{GL}(n)$ leaving invariant a fixed tensor of Young symmetry type π are not necessarily reductive, let alone semi simple Lie groups. The characters $(\!(\lambda)\!)_\pi$ which we study correspond to representations which may be reducible, but not necessarily fully reducible, for $|\pi| > 2$. The full resolution of these issues is beyond the scope of the present introductory study.

In dealing with concrete examples, we have to pass from formal characters to actual characters which implies that there are syzygies and the representations are not in general free modules. Thus, for each π we expect a set of 'standard' characters $(\!(\lambda)\!)_\pi$, together with so-called 'modification rules' for non-standard ones [16, 39]. As mentioned already, a systematic treatment is beyond the scope of the paper, but some hints can be inferred from working through the examples. At the present level of discussion modification rules must be established on a case-by-case basis, and may even depend for each π on different canonical forms of the invariant tensor of symmetry type π. For present purposes, we simply regard the $(\!(\lambda)\!)_\pi$ as a list of formal characters associated with the group H_π. Here and in the following subsections, we take up the case $\pi = \{1^3\}$, and discuss some details of branching, product and modification rules for the formal characters $(\!(\lambda)\!)_{1^3}$. We drop the index $\pi = \{1^3\}$ from now on for brevity and notational clarity. In appendix C further branching and product formulae are given for other cases $\{3\}$, $\{21\}$ but without any analysis of modification rules.

3.1. $\mathsf{H}_{1^3}(n)$ *in dimension* $n = 3$

3.1.1. $\mathsf{SL}(3) \equiv \mathsf{H}_{1^3}(3)$. Consider the case $\pi = \{1^3\}$ corresponding to the existence of an invariant totally antisymmetric third rank tensor η_{ijk} satisfying the conditions:

$$\eta_{ijk} = \eta_{jki} = \eta_{kij} = -\eta_{ikj} = -\eta_{jik} = -\eta_{kji}. \tag{3.1}$$

The requirement that this tensor be invariant under the action of all elements $A \in H_{1^3}(n)$ gives rise to the constraints

$$A : \eta_{ijk} \to A_i^p A_j^q A_k^r \eta_{pqr} = \eta_{ijk} \tag{3.2}$$

with $i, j, k, p, q, r \in \{1, 2, \ldots, n\}$. Here and in what follows Einstein's convention is followed whereby repeated indices, such as p, q and r, are summed over their full range of values, in this case $1, 2, \ldots, n$.

In fact for given n there may be more than one canonical form of the invariant tensor η_{ijk}. For $n = 3$ the canonical form is necessarily defined by $\eta_{ijk} = a\epsilon_{ijk}$ with ϵ_{ijk} the usual third rank totally antisymmetric tensor in a three-dimensional space such that $\epsilon_{123} = 1$. As can be seen from the constraint conditions (3.2) we can take $a = 1$ without loss of generality and set $\eta_{ijk} = \epsilon_{ijk}$. Using this in (3.2) with $n = 3$ just gives $\eta_{123} = \det A = 1$ and up to isomorphism we can immediately make the identification $\mathsf{H}_{1^3}(3) = \mathsf{SL}(3)$. In this case, therefore, the relevant subgroup of $\mathsf{GL}(3)$ is the semisimple Lie group $\mathsf{SL}(3)$. All its representations are fully reducible and we can identify the characters $(\!(\lambda)\!)$ with Schur functions $s_\lambda(x_1, x_2, x_3)$ with constraint $x_1x_2x_3 = 1$ for partitions λ of length less than 3. For other cases certain modification rules are required, to which we shall return later. It is immediate that a similar analysis can be carried out for any $\mathsf{SL}(n) = \mathsf{H}_{1^n}(n)$, which gives a unified character theory for all $\mathsf{SL}(n)$ groups.

3.1.2. $\mathsf{H}_{1^3}(3)$*: Modification rules and* $\mathsf{GL}(3) \supset H_{1^3}(3) = \mathsf{SL}(3)$ *branching rules.* This special case should be trivial because it corresponds to the well-known restriction from $\mathsf{GL}(3)$ to $\mathsf{SL}(3)$. However, even here although the branchings are indeed trivial, the modification rules are somewhat complicated in our formalism.

The branching rule from characters $\{\lambda\}$ of $\mathsf{GL}(3)$ to formal characters $(\!(\mu)\!)$ of $\mathsf{H}_{1^3}(3) = \mathsf{SL}(3)$ takes the form

$$\mathsf{GL}(3) \supset \mathsf{H}_{1^3}(3): \quad \{\lambda\} \to (\!(\lambda/M_{1^3})\!). \tag{3.3}$$

We use the double parentheses $(\!(\lambda)\!)_\pi$ for the formal characters of H_π. If the context is clear we drop the index, which may be replaced by the dimension of the representation. Confusion between integer dimension and partitions cannot occur. The particular series employed here reads

$$M_{1^3} = \{1^3\} \underline{\otimes} M = \{0\} + \{1^3\} + \{2^3\} + \{3^3\} + \{4^3\} + \cdots, \tag{3.4}$$

where it has only been necessary to retain terms of length not greater than 3. This yields the following formal characters, where the subscripts give the dimension of the corresponding representations or more precisely the value of the characters and formal characters at the identity.

$\{\lambda\}_{\dim}$	$(\!(\lambda/M_{1^3})\!)_{\dim}$
$\{0\}_1$	$(\!(0)\!)_1$
$\{1\}_3$	$(\!(1)\!)_3$
$\{11\}_3$	$(\!(11)\!)_3$
$\{111\}_1$	$(\!(111)\!)_0 + (\!(0)\!)_1$
$\{1111\}_0$	$(\!(1111)\!)_{-3} + (\!(1)\!)_3$
$\{2\}_6$	$(\!(2)\!)_6$
$\{21\}_8$	$(\!(21)\!)_8$
$\{211\}_3$	$(\!(211)\!)_0 + (\!(1)\!)_3$
$\{2111\}_0$	$(\!(2111)\!)_{-9} + (\!(2)\!)_6 + (\!(11)\!)_3$
$\{22\}_6$	$(\!(22)\!)_6$
$\{221\}_3$	$(\!(221)\!)_0 + (\!(11)\!)_3$
$\{2211\}_0$	$(\!(2211)\!)_{-8} + (\!(21)\!)_8 + (\!(111)\!)_0$
$\{222\}_1$	$(\!(222)\!)_0 + (\!(111)\!)_0 + (\!(0)\!)_1$
$\{2221\}_0$	$(\!(2221)\!)_0 + (\!(211)\!)_0 + (\!(1111)\!)_{-3} + (\!(1)\!)_3$
$\{2222\}_0$	$(\!(2222)\!)_3 + (\!(2111)\!)_{-9} + (\!(2)\!)_6$
$\{3\}_{10}$	$(\!(3)\!)_{10}$
$\{31\}_{15}$	$(\!(31)\!)_{15}$
$\{311\}_6$	$(\!(311)\!)_0 + (\!(2)\!)_6$
$\{3111\}_0$	$(\!(3111)\!)_{-18} + (\!(3)\!)_{10} + (\!(21)\!)_8$
$\{32\}_{15}$	$(\!(32)\!)_{15}$

(3.5)

For $\ell(\mu) \leqslant 2$, as expected for the branching from $\mathsf{GL}(3)$ to $\mathsf{H}_{1^3}(3) = \mathsf{SL}(3)$, we have $\{\lambda_1, \lambda_2\} \to (\!(\lambda_1, \lambda_2)\!)$. On the other hand, for $\ell(\mu) \geqslant 3$ we require modification rules to interpret the formal characters of $\mathsf{H}_{1^3}(n) = \mathsf{SL}(3)$ in terms of irreducible characters. From the above branchings these must include the following:

$$\begin{aligned}
&((111))_0 = 0 \qquad ((211))_0 = 0 \qquad ((221))_0 = 0\\
&((222))_0 = 0 \qquad ((311))_0 = 0 \qquad ((1111))_{-3} = -((1))_3\\
&((2111))_{-9} = -((2))_6 - ((11))_3 \qquad ((2211))_{-8} = -((21))_8\\
&((2221))_0 = 0 \qquad ((2222))_3 = ((11))_3 \qquad ((3111))_{-18} = -((3))_{10} - ((21))_8.
\end{aligned} \tag{3.6}$$

More generally, for any $((\lambda_1, \lambda_2, \lambda_3, \lambda_4))$ with $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant 0$ the following constitute a complete set of modification rules:

$$\begin{aligned}
&((\lambda_1, \lambda_2, \lambda_3, 0)) = 0 \qquad \text{if} \quad \lambda_3 \geqslant 1\\
&((\lambda_1, \lambda_1, 1, 1)) = -((\lambda_1, \lambda_1 - 1)) \qquad \text{if} \quad \lambda_1 = \lambda_2 \geqslant 1\\
&((\lambda_1, \lambda_2, 1, 1)) = -((\lambda_1, \lambda_2 - 1)) - ((\lambda_1 - 1, \lambda_2)) \qquad \text{if} \quad \lambda_1 > \lambda_2 \geqslant 1\\
&((\lambda_1, \lambda_2, 2, 1)) = 0 \qquad \text{if} \quad \lambda_1 \geqslant \lambda_2 \geqslant 2\\
&((\lambda_1, \lambda_2, 2, 2)) = ((\lambda_1 - 1, \lambda_2 - 1)) \qquad \text{if} \quad \lambda_1 \geqslant \lambda_2 \geqslant 2\\
&((\lambda_1, \lambda_2, \lambda_3, \lambda_4)) = 0 \qquad \text{if} \quad \lambda_3 \geqslant 3.
\end{aligned} \tag{3.7}$$

3.1.3. Product of characters. As a check we can recover the known $\mathsf{SL}(3)$ products through the use of

$$((\mu))((\nu)) = \sum_{\alpha,\beta} (((\mu/\alpha \cdot \{(1^2)\} \underline{\otimes} \beta) \cdot (\nu/\{(1^2)\} \underline{\otimes} \alpha\} \cdot \beta))). \tag{3.8}$$

For example,

$$\begin{aligned}
((22))_6((21))_8 &= (((22) \cdot (21))) + (((22/1) \cdot (21/11))) + (((22/11) \cdot (21/1)))\\
&\quad + (((22/(1 \cdot 11)) \cdot (21/(11 \cdot 1)))) + (((22/22) \cdot (21/2)))\\
&= ((43)) + ((421)) + ((331)) + ((322)) + ((3211)) + ((2221))\\
&\quad + 2((31)) + 2((22)) + 3((211)) + ((1111)) + 2((1))\\
&= ((43))_{24} + ((31))_{15} + ((22))_6 + ((1))_3
\end{aligned} \tag{3.9}$$

where the modification rules have been used in the last step. The result agrees with what we obtain by going up (trivially) from $\mathsf{H}_{1^3}(3) = \mathsf{SL}(3)$ to $\mathsf{GL}(3)$, carrying out the product in $\mathsf{GL}(3)$ and branching to $\mathsf{SL}(3)$ by throwing away columns of length greater than 3.

$$\begin{aligned}
((22))_6((21))_8 &= \{22\}\{21\} = \{43\} + \{421\} + \{331\} + \{322\}\\
&= \{43\} + \{31\} + \{22\} + \{1\}\\
&= ((43))_{24} + ((31))_{15} + ((22))_6 + ((1))_3.
\end{aligned} \tag{3.10}$$

3.2. $\mathsf{H}_{1^3}(4)$: *branching rules, modification rules and products of characters*

3.2.1. Matrix realization. Turning to the case $n = 4$, the canonical form of η is such that with a suitable scaling

$$\eta_{ijk} = \begin{cases} \epsilon_{abc} & \text{for} \quad (i, j, k) = (a, b, c) \quad \text{with} \quad a, b, c \in \{1, 2, 3\};\\ 0 & \text{otherwise.} \end{cases} \tag{3.11}$$

Thus, the constraints (3.2) reduce to

$$A_i^a A_j^b A_k^c \epsilon_{abc} = \eta_{ijk}, \tag{3.12}$$

so that

$$A_4^a A_4^b A_4^c \epsilon_{abc} = \eta_{444} = 0, \tag{3.13}$$

$$A_p^a A_4^b A_4^c \epsilon_{abc} = \eta_{p44} = 0, \tag{3.14}$$

$$A_p^a A_q^b A_4^c \epsilon_{abc} = \eta_{pq4} = 0, \tag{3.15}$$

$$A_p^a A_q^b A_r^c \epsilon_{abc} = \eta_{pqr} = \epsilon_{pqr}, \tag{3.16}$$

with $p, q, r \in \{1, 2, 3\}$. The first two constraints (3.13) and (3.14) are satisfied automatically because of the antisymmetry of ϵ_{abc}. The fourth constraint (3.16) gives $\det B = 1$ where B is the 3×3 submatrix of A such that $B_a^b = A_a^b$ for all $a, b \in \{1, 2, 3\}$. With this notation, (3.16) becomes

$$B_p^a B_q^b B_r^c \epsilon_{abc} = \epsilon_{pqr}, \tag{3.17}$$

so that

$$B^{-1}{}^r{}_s \epsilon_{pqr} = B_p^a B_q^b B^{-1}{}^r{}_s B_r^c \epsilon_{abc} = B_p^a B_q^b \epsilon_{abs}. \tag{3.18}$$

Using this in (3.15) then gives

$$A_4^c B^{-1}{}^r{}_c \epsilon_{pqr} = 0. \tag{3.19}$$

Since this is true for all $p, q \in \{1, 2, 3\}$ it follows that

$$A_4^c B^{-1}{}^r{}_c = 0 \tag{3.20}$$

for all $r \in \{1, 2, 3\}$. Hence,

$$A_4^c B^{-1}{}^r{}_c B_r^d = A_4^d = 0 \tag{3.21}$$

for all $d \in \{1, 2, 3\}$. Thus, A necessarily takes the form

$$A = \begin{pmatrix} B & D \\ 0 & C \end{pmatrix} \qquad \text{with} \quad \det B = 1, \quad \det C \neq 0. \tag{3.22}$$

where D is an arbitrary 3×1 matrix, C is a non-zero 1×1 matrix and 0 signifies a 1×3 zero matrix in accordance with (3.21). It follows that $\mathsf{H}_{1^3}(4)$ is subgroup of $\mathsf{GL}(4)$ consisting of non-singular 4×4 matrices of the form (3.22). It should be noted that this is an example of an affine group. It is neither semisimple nor reductive, as can be seen from the block triangular form of the defining four-dimensional reducible, but indecomposable, representation (3.22). It contains the reductive group $\mathsf{SL}(3) \times \mathsf{GL}(1)$ as a proper subgroup. In fact, we have shown that the formal character theory can be extended to non-semisimple groups in a straightforward manner. The presented example $\mathsf{H}_{1^3}(4)$ contains after the further branching $\mathsf{SL}(3) \downarrow \mathsf{SO}(3)$ the group of motions of a rigid body and has as such potential applications in robotics, etc.

3.2.2. Branching rules. First note that the inequivalent finite-dimensional rational irreducible representations of $\mathsf{GL}(4)$ have characters $\varepsilon^r\{\lambda\} = \varepsilon^r s_\lambda(x)$ with r an integer and $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ a partition of length $\ell(\lambda) \leqslant 4$. The parameters $x = (x_1, x_2, x_3, x_4)$ are the eigenvalues of the group elements $A \in GL(4)$ and ε is the character of the representation $A \mapsto \det(A)$, so that $\varepsilon = s_{1^4}(x) = \{1^4\} = x_1 x_2 x_3 x_4$.

The branching rule from characters $\{\lambda\}$ of $\mathsf{GL}(4)$ to formal characters $(\!(\mu)\!)$ of $\mathsf{H}_{1^3}(4)$ takes the form

$$\mathsf{GL}(4) \supset \mathsf{H}_{1^3}(4): \quad \{\lambda\} \to (\!(\lambda/M_{1^3})\!) \tag{3.23}$$

with

$$M_{1^3} = \{1^3\} \underline{\otimes} M = \{0\} + \{1^3\} + \{2^3\} + \{3^3\} + \{4^3\} + \cdots, \tag{3.24}$$

where it has only been necessary to retain terms of length not greater than 4. This yields the following data, where the subscripts give the dimension of the corresponding representations or more precisely the value of the characters and formal characters at the identity.

$\{\lambda\}_{\mathrm{dim}}$	$\langle\!\langle \lambda/M_{1^3} \rangle\!\rangle_{\mathrm{dim}}$
$\{0\}_1$	$\langle\!\langle 0 \rangle\!\rangle_1$
$\{1\}_4$	$\langle\!\langle 1 \rangle\!\rangle_4$
$\{11\}_6$	$\langle\!\langle 11 \rangle\!\rangle_6$
$\{111\}_4$	$\langle\!\langle 111 \rangle\!\rangle_3 + \langle\!\langle 0 \rangle\!\rangle_1$
$\{1111\}_1$	$\langle\!\langle 1111 \rangle\!\rangle_{-3} + \langle\!\langle 1 \rangle\!\rangle_4$
$\{2\}_{10}$	$\langle\!\langle 2 \rangle\!\rangle_{10}$
$\{21\}_{20}$	$\langle\!\langle 21 \rangle\!\rangle_{20}$
$\{211\}_{15}$	$\langle\!\langle 211 \rangle\!\rangle_{11} + \langle\!\langle 1 \rangle\!\rangle_4$
$\{2111\}_4$	$\langle\!\langle 2111 \rangle\!\rangle_{-12} + \langle\!\langle 2 \rangle\!\rangle_{10} + \langle\!\langle 11 \rangle\!\rangle_6$
$\{22\}_{20}$	$\langle\!\langle 22 \rangle\!\rangle_{20}$
$\{221\}_{20}$	$\langle\!\langle 221 \rangle\!\rangle_{14} + \langle\!\langle 11 \rangle\!\rangle_6$
$\{2211\}_6$	$\langle\!\langle 2211 \rangle\!\rangle_{-17} + \langle\!\langle 21 \rangle\!\rangle_{20} + \langle\!\langle 111 \rangle\!\rangle_3$
$\{222\}_{10}$	$\langle\!\langle 222 \rangle\!\rangle_6 + \langle\!\langle 111 \rangle\!\rangle_3 + \langle\!\langle 0 \rangle\!\rangle_1$
$\{2221\}_4$	$\langle\!\langle 2221 \rangle\!\rangle_{-8} + \langle\!\langle 211 \rangle\!\rangle_{11} + \langle\!\langle 1111 \rangle\!\rangle_{-3} + \langle\!\langle 1 \rangle\!\rangle_4$
$\{2222\}_1$	$\langle\!\langle 2222 \rangle\!\rangle_3 + \langle\!\langle 2111 \rangle\!\rangle_{-12} + \langle\!\langle 2 \rangle\!\rangle_{10}$
$\{3\}_{20}$	$\langle\!\langle 3 \rangle\!\rangle_{20}$
$\{31\}_{45}$	$\langle\!\langle 31 \rangle\!\rangle_{45}$
$\{311\}_{36}$	$\langle\!\langle 311 \rangle\!\rangle_{26} + \langle\!\langle 2 \rangle\!\rangle_{10}$
$\{3111\}_{10}$	$\langle\!\langle 3111 \rangle\!\rangle_{-30} + \langle\!\langle 3 \rangle\!\rangle_{20} + \langle\!\langle 21 \rangle\!\rangle_{20}$
$\{32\}_{60}$	$\langle\!\langle 32 \rangle\!\rangle_{60}$

(3.25)

3.2.3. Aspects of $\mathsf{H}_{1^3}(4)$ *modification rules.* To interpret the formal characters of $\mathsf{H}_{1^3}(4)$ it is necessary to apply modification rules. These are required for all $\langle\!\langle \mu \rangle\!\rangle$ with μ of length $\ell(\mu) = 4$. In these branchings, these formal characters can only arise in cases for which λ also has length 4. However, as pointed out above $\{1^4\} = \varepsilon$ is the character of the one-dimensional determinant representation of $\mathsf{GL}(4)$. It follows that

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \varepsilon^{\lambda_4}\{\lambda_1 - \lambda_4, \lambda_2 - \lambda_4, \lambda_3 - \lambda_4, 0\}. \tag{3.26}$$

Applying this to the tabulated identities gives

$$\begin{aligned}
\langle\!\langle 1111 \rangle\!\rangle_{-3} &= \{1^4\}_1 - \langle\!\langle 1 \rangle\!\rangle_4 \\
&= \varepsilon\langle\!\langle 0 \rangle\!\rangle_1 - \langle\!\langle 1 \rangle\!\rangle_4 \\
\langle\!\langle 2111 \rangle\!\rangle_{-12} &= \{2111\}_4 - \langle\!\langle 2 \rangle\!\rangle_{10} - \langle\!\langle 11 \rangle\!\rangle_6 \\
&= \varepsilon\langle\!\langle 1 \rangle\!\rangle_4 - \langle\!\langle 2 \rangle\!\rangle_{10} - \langle\!\langle 11 \rangle\!\rangle_6 \\
\langle\!\langle 2211 \rangle\!\rangle_{-17} &= \{2211\}_6 - \langle\!\langle 21 \rangle\!\rangle_{20} - \langle\!\langle 111 \rangle\!\rangle_3 \\
&= \varepsilon\langle\!\langle 11 \rangle\!\rangle_6 - \langle\!\langle 21 \rangle\!\rangle_{20} - \langle\!\langle 111 \rangle\!\rangle_3 \\
\langle\!\langle 2221 \rangle\!\rangle_{-8} &= \{2221\}_4 - \langle\!\langle 211 \rangle\!\rangle_{11} - \langle\!\langle 1111 \rangle\!\rangle_{-3} - \langle\!\langle 1 \rangle\!\rangle_4 \\
&= \varepsilon\{111\}_4 - \langle\!\langle 211 \rangle\!\rangle_{11} - \varepsilon\langle\!\langle 0 \rangle\!\rangle_1 + \langle\!\langle 1 \rangle\!\rangle_4 - \langle\!\langle 1 \rangle\!\rangle_4
\end{aligned}$$

$$
\begin{aligned}
&= \varepsilon(\langle\!\langle 111\rangle\!\rangle_3 + \langle\!\langle 0\rangle\!\rangle_1) - \langle\!\langle 211\rangle\!\rangle_{11} - \varepsilon\langle\!\langle 0\rangle\!\rangle_1 \\
&= \varepsilon\langle\!\langle 111\rangle\!\rangle_3 - \langle\!\langle 211\rangle\!\rangle_{11} \\
\langle\!\langle 2222\rangle\!\rangle_3 &= \{2222\}_1 - (\langle\!\langle 2111\rangle\!\rangle_{-12} - \langle\!\langle 2\rangle\!\rangle_{10} \\
&= \varepsilon^2\{0\}_1 - \varepsilon\langle\!\langle 1\rangle\!\rangle_4 + \langle\!\langle 2\rangle\!\rangle_{10} + \langle\!\langle 11\rangle\!\rangle_6 - \langle\!\langle 2\rangle\!\rangle_{10} \\
&= \varepsilon^2\langle\!\langle 0\rangle\!\rangle_1 - \varepsilon\langle\!\langle 1\rangle\!\rangle_4 + \langle\!\langle 11\rangle\!\rangle_6 \\
\langle\!\langle 3111\rangle\!\rangle_{-30} &= \{3111\}_{10} - \langle\!\langle 3\rangle\!\rangle_{20} - \langle\!\langle 21\rangle\!\rangle_{20} \\
&= \varepsilon\langle\!\langle 2\rangle\!\rangle_{10} - \langle\!\langle 3\rangle\!\rangle_{20} - \langle\!\langle 21\rangle\!\rangle_{20}.
\end{aligned}
\tag{3.27}
$$

This gives a collection of modification rules to be applied in the case $\langle\!\langle \mu\rangle\!\rangle$ with $\ell(\mu) = 4$. Of course, one would like to have a complete set of modification rules, including those appropriate to $\langle\!\langle \mu\rangle\!\rangle$ with $\ell(\mu) > 4$.

3.2.4. Product of characters. We can exploit this to analyse products of characters. For example,

$$
\begin{aligned}
\langle\!\langle 2\rangle\!\rangle_{10}\langle\!\langle 111\rangle\!\rangle_3 &= \langle\!\langle 311\rangle\!\rangle_{26} + \langle\!\langle 2111\rangle\!\rangle_{-12} + \langle\!\langle 2\rangle\!\rangle_{10} + \langle\!\langle 11\rangle\!\rangle_6 \\
&= \langle\!\langle 311\rangle\!\rangle_{26} + (\varepsilon\langle\!\langle 1\rangle\!\rangle_4 - \langle\!\langle 2\rangle\!\rangle_{10} - \langle\!\langle 11\rangle\!\rangle_6) + \langle\!\langle 2\rangle\!\rangle_{10} + \langle\!\langle 11\rangle\!\rangle_6 \\
&= \langle\!\langle 311\rangle\!\rangle_{26} + \varepsilon\langle\!\langle 1\rangle\!\rangle_4.
\end{aligned}
\tag{3.28}
$$

Further examples of products of $\mathsf{H}_{1^3}(4)$ characters are given in appendix C.3.

Clearly by proceeding in this way we can build up branching rules, modification rules and product rules for characters for each H_π on a case-by-case basis (including for example obtaining each $\mathsf{SL}(n)$ as a subgroup $\mathsf{H}_{1^n}(n)$ of $\mathsf{GL}(n)$ in a unified framework). However, we now close this heuristic section and pause for an introduction to Hopf algebra methods, so as to start to develop these results into a general theory.

4. Hopf algebraic analysis

4.1. Notions and definitions

In section 2.3, the series $M_\pi = \{\pi\} \underline{\otimes} M$ and associated formal characters for non-classical subgroups were derived from straightforward generalizations of the known combinatorial route to the $\mathsf{GL}(n)$ subgroups $\mathsf{GL}(n-1)$, $\mathsf{O}(n)$ and $\mathsf{Sp}(n)$. The aim of the present section is to embed the combinatorial proofs, given above and in the appendix, into the discussion of branching and product rules, in the context of the Hopf algebraic structure of, and the associated cohomological framework for, symmetric functions as developed in [11]. The Hopf algebraic aspects are well known in the combinatorial literature [13, 43, 42, 37, 36]

In [11], these branching rules were studied from the point of view of cliffordization (multiplicative deformations, compare equation (4.17) with equation (4.19)) of the algebraic structure, and initial steps were taken towards cohomological classifications. Here we give in outline the basic setting needed to clarify the relationship between series-induced branching rules and associative deformations (2-cocycles). A detailed development of the Hopf algebra cohomology involved was given in [4]. This algebraic structure has been used in a similar manner in [11]. In section 4.2, standard rules for plethysm distributivity are introduced, transcribed into the language of products and coproducts of symmetric functions. Finally, section 4.4 leads into the presentation of a very general class of 2-cochains, of which the present π-branching rules provide examples which are also 2-cocycles. The abstract development is complemented in appendix A by a brief introduction to a systematic diagrammatical approach using tangles.

The basic structure of interest is the outer Hopf algebra of the ring of symmetric functions Λ. Given the canonical Schur scalar product $\langle\cdot\mid\cdot\rangle:\Lambda\otimes\Lambda\to\mathbb{Z}$ which makes the Schur functions orthonormal,

$$\langle s_\lambda|s_\mu\rangle=\delta_{\lambda\mu}, \tag{4.1}$$

we define the outer coproduct of any symmetric function F, $\Delta(F)$, by duality, that is

$$\langle\Delta(F)\mid G\otimes H\rangle:=\langle F\mid G\cdot H\rangle \tag{4.2}$$

for any symmetric functions G and H, where $\cdot$ is outer multiplication. On the basis of Schur functions this yields using (2.39)

$$\Delta(s_\pi)=\sum_{\xi,\eta}C^\pi_{\xi\eta}s_\xi\otimes s_\eta. \tag{4.3}$$

There is a similar inner coproduct which dualizes inner multiplication of symmetric functions,

$$\langle\delta(F)\mid G\otimes H\rangle:=\langle F\mid G\star H\rangle \tag{4.4}$$

for which the structure constants in the Schur function basis are therefore the structure constants of inner multiplication,

$$\delta(s_\pi)=\sum_{\xi,\eta}\Gamma^\pi_{\xi\eta}s_\xi\otimes s_\eta. \tag{4.5}$$

In both cases we can adapt the Sweedler convention of affixing bracketed subscripts to denote the list of parts of the coproducts. To distinguish different coproducts we employ the Brouder–Schmitt convention [6] using (1), (2) for the outer coproduct Δ and [1, 2] for the inner coproduct δ:

$$\Delta(F)=\sum_{(F)}F_{(1)}\otimes F_{(2)}=F_{(1)}\otimes F_{(2)}\qquad \delta(F)=\sum_{[F]}F_{[1]}\otimes F_{[2]}=F_{[1]}\otimes F_{[2]}. \tag{4.6}$$

For later use we define *proper cuts* of the outer coproduct to be the sum of those tensors which do not contain the unit. The proper cut part is denoted by primes,

$$\Delta'(F)=F'_{(1)}\otimes F'_{(2)}=\Delta(F)-F\otimes 1-1\otimes F. \tag{4.7}$$

Outer multiplication is of course nothing but pointwise multiplication of symmetric polynomials. Inner multiplication is most easily described in the Schur function basis and is related to products of characters of the symmetric group (non-zero only for partitions of the same rank).

Example 4.9. Using (4.3) gives, for example,

$$\begin{aligned}
&\Delta(\{0\})=\{0\}\otimes\{0\},\\
&\Delta(\{1\})=\{1\}\otimes\{0\}+\{0\}\otimes\{1\},\\
&\Delta(\{2\})=\{2\}\otimes\{0\}+\{1\}\otimes\{1\}+\{0\}\otimes\{2\},\\
&\Delta(\{1^2\})=\{1^2\}\otimes\{0\}+\{1\}\otimes\{1\}+\{0\}\otimes\{1^2\},\\
&\Delta(\{3\})=\{3\}\otimes\{0\}+\{2\}\otimes\{1\}+\{1\}\otimes\{2\}+\{0\}\otimes\{3\},\\
&\Delta(\{21\})=\{21\}\otimes\{0\}+\{2\}\otimes\{1\}+\{1^2\}\otimes\{1\}+\{1\}\otimes\{2\}+\{1\}\otimes\{1^2\}+\{0\}\otimes\{21\},\\
&\Delta(\{1^3\})=\{1^3\}\otimes\{0\}+\{1^2\}\otimes\{1\}+\{1\}\otimes\{1^2\}+\{0\}\otimes\{1^3\},\\
&\Delta(\{2^2\})=\{2^2\}\otimes\{0\}+\{21\}\otimes\{1\}+\{2\}\otimes\{2\}+\{1^2\}\otimes\{1^2\}+\{1\}\otimes\{21\}+\{0\}\otimes\{2^2\};
\end{aligned} \tag{4.8}$$

and (4.5) yields

$$\begin{aligned}
&\delta(\{0\}) = \{0\} \otimes \{0\}, \qquad \delta(\{1\}) = \{1\} \otimes \{1\},\\
&\delta(\{2\}) = \{2\} \otimes \{2\} + \{1^2\} \otimes \{1^2\}, \qquad \delta(\{1^2\}) = \{2\} \otimes \{1^2\} + \{1^2\} \otimes \{2\},\\
&\delta(\{3\}) = \{3\} \otimes \{3\} + \{21\} \otimes \{21\} + \{1^3\} \otimes \{1^3\},\\
&\delta(\{21\}) = \{3\} \otimes \{21\} + \{21\} \otimes \{3\} + \{21\} \otimes \{21\} + \{21\} \otimes \{1^3\} + \{1^3\} \otimes \{21\},\\
&\delta(\{1^3\}) = \{3\} \otimes \{1^3\} + \{21\} \otimes \{21\} + \{1^3\} \otimes \{3\},\\
&\delta(\{2^2\}) = \{4\} \otimes \{2^2\} + \{31\} \otimes \{31\} + \{31\} \otimes \{21^2\} + \{2^2\} \otimes \{4\} + \{2^2\} \otimes \{2^2\},\\
&\qquad + \{2^2\} \otimes \{1^4\} + \{21^2\} \otimes \{31\} + \{21^2\} \otimes \{21^2\} + \{1^4\} \otimes \{2^2\},
\end{aligned} \tag{4.9}$$

while the proper cut outer coproduct reads

$$\begin{aligned}
&\Delta'(\{2\}) = \{1\} \otimes \{1\}, \qquad \Delta'(\{1^2\}) = \{1\} \otimes \{1\},\\
&\Delta'(\{3\}) = \{2\} \otimes \{1\} + \{1\} \otimes \{2\},\\
&\Delta'(\{21\}) = \{2\} \otimes \{1\} + \{1^2\} \otimes \{1\} + \{1\} \otimes \{2\} + \{1\} \otimes \{1^2\},\\
&\Delta'(\{1^3\}) = \{1^2\} \otimes \{1\} + \{1\} \otimes \{1^2\},\\
&\Delta'(\{2^2\}) = \{21\} \otimes \{1\} + \{2\} \otimes \{2\} + \{1^2\} \otimes \{1^2\} + \{1\} \otimes \{21\}.
\end{aligned} \tag{4.10}$$

Note that $\{2\}$ and $\{1^2\}$ have the same proper cut parts.

The coalgebra structure provided by the outer coproduct extends, due to Frobenius reciprocity, to an outer Hopf algebra. The remaining ingredients are the antipode antihomomorphism map,

$$\mathsf{S}(\{\mu\}) = (-1)^{|\mu|}\{\mu'\}, \tag{4.11}$$

where μ' is the partition conjugate to μ, and the counit character which is given by

$$\epsilon(\{\mu\}) = \delta_{\mu 0}. \tag{4.12}$$

The inner product and inner coproduct do not form a Hopf algebra, but only inner convolution algebras (see [11]).

Group branchings have a very neat description in terms of Schur function series and the language of Hopf algebra cohomology [35, 4]. A linear form $\phi : \Lambda \to \mathbb{Z}$ acting on the ring of symmetric functions Λ with values in $\mathbb{Z}$ is called a 1-cochain. Using such 1-cochains ϕ, the branching operator obtained by skewing with a series $/\Phi$ reads in Hopf algebraic terms

$$s_{\lambda/\Phi} = \{\lambda\}/\Phi = (\phi \otimes \mathrm{Id})\Delta(s_\lambda), \tag{4.13}$$

where for given Φ the 1-cochain ϕ is defined by

$$\phi(s_\mu) = \langle \Phi | s_\mu \rangle. \tag{4.14}$$

The operator $/\Phi$ is called a branching operator, recalling the group branchings of (2.16, 2.17, 2.18). We adopt the convention that 1-cochains are denoted by lowercase letters, and the corresponding Schur function series by uppercase letters. Given a $\mathsf{GL}(n)$ character, a branching process using an appropriate branching operator yields Schur functions describing the formal characters of the appropriate $\mathsf{GL}(n)$ subgroup.

Extending from linear forms (1-cochains) to multilinear forms of n arguments, we define n-cochains $c_n : \Lambda^{\otimes n} \rightarrow \mathbb{Z}$. Having a Hopf algebra Λ, we can define on all endomorphisms $f, g : \Lambda \rightarrow \Lambda$ the convolution product $* : \mathrm{End}\,\Lambda \times \mathrm{End}\,\Lambda \rightarrow \mathrm{End}\,\Lambda$, $(f * g)(X) = f(X_{(1)})g(X_{(2)})$. This product can be generalized in a straightforward manner to n-cochains with multiplication taken in $\mathbb{Z}$. We define a multiplicatively written coboundary operator ∂_n mapping n-cochains c_n to $(n+1)$-cochains c_{n+1} by means of

$$
\begin{aligned}
c_{n+1}(X_0, \ldots, X_n) &= \partial^i c_n(X_0, \ldots, X_n) \\
&= \begin{cases} \epsilon(X_0)c_n(X_1, \ldots, X_n) & i = 0 \\ c_n(X_0, \ldots, X_{i-1} \cdot X_i, \ldots, X_n) & i \in \{1, \ldots, n\} \\ c_n(X_0, \ldots, X_{n-1})\epsilon(X_n) & i = n+1 \end{cases}
\end{aligned}
\tag{4.15}
$$

and

$$
\partial_n c_n = \partial^0 c_n * \partial^1 c_n^{-1} * \ldots * \partial^{n+1} c_n^{\pm 1},
\tag{4.16}
$$

with alternating exponents ± 1. If $\partial_n c_n = \epsilon^{n+1}$ then c_n is closed, and if $\partial_n c_n = c_{n+1}$ then c_{n+1} is exact.

Returning to the branching process associated with the series Φ and the operator $/\Phi$, with the help of the inverse series Φ^{-1} we have in the same way the operator $/\Phi^{-1}$, which is in fact the inverse of $/\Phi$ under the convolution product as is easily checked. Finally, we can define a new product

$$
(\!(A)\!) \cdot_\phi (\!(B)\!) = (\!((A/\Phi^{-1} \cdot B/\Phi^{-1})/\Phi)\!).
\tag{4.17}
$$

It is remarkable that one can derive a 2-cocycle $(\partial\phi^{-1})$ from the linear form ϕ^{-1}, which allows one to reinterpret this process as a cliffordization or twist (see also appendix A). Using (4.13), (4.14) and Sweedler indices (see above), we can rewrite this deformed product as

$$
\begin{aligned}
(\!(A)\!) \cdot_\phi (\!(B)\!) &= (\phi \otimes \mathrm{Id})\Delta(\phi^{-1}(A_{(1)})\phi^{-1}(B_{(1)})A_{(2)} \cdot B_{(2)}) \\
&= \phi^{-1}(A_{(1)})\phi^{-1}(B_{(1)})\phi(A_{(2)} \cdot B_{(2)})A_{(3)} \cdot B_{(3)}.
\end{aligned}
\tag{4.18}
$$

In the second line the homomorphism axiom of the outer Hopf algebra was employed to convert the coproduct of a product into the product of coproducts, thereby introducing new Sweedler indices. Then identifying the ϕ-terms using (4.16) as a 2-cocycle $\partial\phi^{-1}$, where the 2-cocycle $(\partial\phi^{-1}) := \partial_1 c_1$ with $c_1 = \phi^{-1}$, and a suitable change of Sweedler indices, this product can be rewritten in the form

$$
A \cdot_\phi B = (\partial\phi^{-1})(A_{(1)}, B_{(1)})A_{(2)} \cdot B_{(2)}.
\tag{4.19}
$$

This can be seen by using the convolution product (4.16) in the case $n = 1$ and $c_n = \phi^{-1}$, together with (4.15) with $X_0 = A$ and $X_1 = B$. This gives

$$
\begin{aligned}
(\partial\phi^{-1})(A, B) &= \epsilon(A_{(1)})\phi^{-1}(B_{(1)})\phi(A_{(2)} \cdot B_{(2)})\phi^{-1}(A_{(3)})\epsilon(B_{(3)}) \\
&= \phi^{-1}(A_{(1)})\phi^{-1}(B_{(1)})\phi(A_{(2)} \cdot B_{(2)}),
\end{aligned}
\tag{4.20}
$$

the required result that the left-hand sides of (4.18) and (4.19) define the same product. It should be noted that the condition to be closed, that is $\partial\phi = \epsilon^2$, implies $A \cdot_\phi B = A \cdot B$ reflecting the homomorphism property of ϕ, that is $\phi(A \cdot B) = \phi(A)\phi(B)$.

The algebra with respect to the ϕ-deformed product is in general no longer Hopf, but forms a comodule algebra. In general the branching operator $/\Phi$ is not an algebra homomorphism, and the lack of being homomorphic controls the new branchings. In [11], it was investigated in which way Hopf algebra cohomology helps to classify branching operators. The result was roughly that one distinguishes 1-cochains which are closed and those that are not closed. The first class yields branching operators $/\Phi$ which are homomorphisms, corresponding to so-called group like series Φ, which are such that

$$(s_\lambda \cdot s_\mu)/\Phi = s_\lambda/\Phi \cdot s_\mu/\Phi. \tag{4.21}$$

This is related to the fact that in these cases

$$\Delta(\Phi) = \sum_{(\Phi)} \Phi_{(1)} \otimes \Phi_{(2)} = \Phi \otimes \Phi. \tag{4.22}$$

Group like series are of the general form

$$\Phi = \prod_{i>0} (1 - f(x_i))^\gamma \tag{4.23}$$

where f is an arbitrary polynomial and γ is in principle arbitrary (invoking the obvious extension of binomial coefficients in the complex case). The epithet 'group like' stems from Hopf algebra theory, where elements with a coproduct $\Delta(g) = g \otimes g$, just doubling the element, are called group like. The second class of series is more interesting. Such series no longer define homomorphisms but one can introduce a 2-cocycle, derived from the 1-cochain, which describes the deviation from being a homomorphism. In this way, Hopf algebra cohomology formalizes the classification of these branching rules. Further analysis shows in fact that a cliffordization, that is a Drinfeld twist, underlies this mechanism. Thus as mentioned, the structure of symmetric functions of orthogonal and symplectic type is embedded in deformation theory in the Hopf algebra context. The reader interested in further details is kindly referred to [11].

4.2. π-Branchings and products of π-characters

Having introduced the requisite Hopf algebraic structure for the understanding of branching rules, we now proceed to revisit the combinatorial analysis of proposition 2.3. Firstly we invoke the following properties of plethysms, transcribed into the present setting.

Lemma 4.10. *For any symmetric functions A, B and C we have*

$$A \underline{\otimes} (B \pm C) = A \underline{\otimes} B \pm A \underline{\otimes} C, \qquad \text{right distributivity,} \tag{4.24}$$

$$(A + B) \underline{\otimes} C = \sum A \underline{\otimes} C_{(1)} \cdot B \underline{\otimes} C_{(2)} \qquad \text{left binomial expansion (i),} \tag{4.25}$$

$$(A - B) \underline{\otimes} C = \sum A \underline{\otimes} C_{(1)} \cdot B \underline{\otimes} \mathsf{S}(C_{(2)}) \qquad \text{left binomial expansion (ii),} \tag{4.26}$$

$$A \underline{\otimes} (B \cdot C) = (A \underline{\otimes} B) \cdot (A \underline{\otimes} C), \qquad \text{right homomorphism,} \tag{4.27}$$

$$(A \cdot B) \underline{\otimes} C = \sum A \underline{\otimes} C_{[1]} \cdot B \underline{\otimes} C_{[2]}, \qquad \text{left homomorphism expansion,} \tag{4.28}$$

$$A \underline{\otimes} (B \underline{\otimes} C) = (A \underline{\otimes} B) \underline{\otimes} C, \qquad \text{associativity.} \tag{4.29}$$

These are standard relations given first by Littlewood [26], but see also [27, 29, 28, 32], or more recently, [8, 7] as well as [39], and the web site of Brian Wybourne[10].

[10] url: http://www.phys.uni.torun.pl/˜bgw/.

As pointed out previously, it is sometimes convenient to use an alternative rather suggestive notation for plethysms, by writing $s_\mu[s_\lambda] = \{\lambda\} \underline{\otimes} \{\mu\} \equiv \{\lambda\}^{\underline{\otimes}\{\mu\}}$. This latter notation will be used in our further development, including that of the following commutativity condition applying to all pairs of partitions λ and μ:

$$\Delta(\{\lambda\}^{\underline{\otimes}\{\mu\}}) = (\Delta(\{\lambda\}))^{\underline{\otimes}\{\mu\}}. \tag{4.30}$$

This may be seen by expressing the left-hand side of (4.30) in terms of indeterminates $z = (x, y)$, and then noting that

$$\Delta(\{\lambda\}^{\underline{\otimes}\{\mu\}})(z) = (\{\lambda\}^{\underline{\otimes}\{\mu\}})(x, y) = s_\mu[s_\lambda](x, y) = s_\mu[\Delta(\{\lambda\}](z) = (\Delta(\{\lambda\})(z))^{\underline{\otimes}\{\mu\}}, \tag{4.31}$$

where this is just the right-hand side of (4.30).

Alternatively, setting $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_n)$ with $z = (x, y)$ the above outer coproduct of a plethysm (the left-hand side) or plethysm of an outer coproduct (the right-hand side) may both be interpreted in terms of branchings from $\mathsf{GL}(m+n)$ to $\mathsf{GL}(m) \times \mathsf{GL}(n)$. It is immaterial at what stage the plethysm evaluation is performed. In other words, there is an appropriate commuting diagram:

$$\begin{array}{ccc} \mathsf{GL}(m+n) & \xrightarrow{\underline{\otimes}\{\mu\}} & \mathsf{GL}(m+n) \\ \downarrow \Delta & & \downarrow \Delta \\ \mathsf{GL}(m) \otimes \mathsf{GL}(n) & \xrightarrow{\underline{\otimes}\{\mu\}} & \mathsf{GL}(m) \otimes \mathsf{GL}(n) \end{array} \tag{4.32}$$

Or in terms of elements:

$$\begin{array}{ccc} \{\lambda\} & \xrightarrow{\underline{\otimes}\{\mu\}} & \{\lambda\}\underline{\otimes}\{\mu\} \\ \downarrow \Delta & & \downarrow \Delta \\ \Delta(\{\lambda\}) & \xrightarrow{\underline{\otimes}\{\mu\}} & \Delta(\{\lambda\}\underline{\otimes}\{\mu\}) = (\Delta(\{\lambda\}))\underline{\otimes}\{\mu\} \end{array} \tag{4.33}$$

Using this observation, we may explicitly evaluate the coproduct of a plethysm as follows:

Lemma 4.11. *The coproduct of the plethysm* $\{\lambda\} \underline{\otimes} \{\mu\}$ *is given by*

$$\Delta(\{\lambda\} \underline{\otimes} \{\mu\}) = \sum_{[\mu_{(1)}]} \cdots \sum_{[\mu_{(N^\lambda)}]} \sum_{(\mu)^{N^\lambda}} \prod_{k=1}^{N^\lambda} (\{\lambda_{(1)k}\} \underline{\otimes} \{\mu_{(k)[1]}\}) \otimes (\{\lambda_{(2)k}\} \underline{\otimes} \{\mu_{(k)[2]}\}) \tag{4.34}$$

where N^λ is the number of terms in the outer coproduct of $\{\lambda\}$. The summations are a variety of Sweedler sums: each $\sum_{[\mu]}$ denotes an inner coproduct Sweedler sum, while $\sum_{(\mu)^N}$ denotes a multiple outer coproduct Sweedler sum.

Proof.

$$\begin{aligned} \Delta(\{\lambda\} \underline{\otimes} \{\mu\}) = (\Delta(\{\lambda\}))^{\underline{\otimes}\{\mu\}} &= \left(\sum_{k=1}^{N^\lambda} \{\lambda_{(1)k}\} \otimes \{\lambda_{(2)k}\} \right)^{\underline{\otimes}\{\mu\}} \\ &= \sum_{(\mu)^{N^\lambda}} \prod_{k=1}^{N^\lambda} (\{\lambda_{(1)k}\} \otimes \{\lambda_{(2)k}\})^{\underline{\otimes}\{\mu_{(k)}\}} \end{aligned}$$

$$= \sum_{[\mu_{(1)}]} \cdots \sum_{[\mu_{(N^\lambda)}]} \sum_{(\mu)^{N^\lambda}} \prod_{k=1}^{N^\lambda} \{\lambda_{(1)k}\}^{\underline{\otimes}\{\mu_{(k)[1]}\}} \otimes \{\lambda_{(2)k}\}^{\underline{\otimes}\{\mu_{(k)[2]}\}}. \tag{4.35}$$

□

Generalizations of the above will be required in which multiple outer coproduct Sweedler sums of the type $\sum_{(\Phi)^N}$ occur, where N is the number of Sweedler sums and Φ is a Schur function series. In the following N will often be given, as here, by the number N^λ of terms in the outer coproduct of some λ.

We will need the following corollary for the general formula of the coproduct of the composition of two series. We use 'virtual representations' [22], which are necessary to turn the monoid of representations into an Abelian group in the same style as Grothendieck. The notion $-\{\mu\}$ is shorthand for $0 - \{\mu\}$. A proper definition would include equivalence classes of pairs of representations under the relation $\{\mu\} + (-\{\mu\}) = 0$; this is not displayed here since we need only basic facts about $-\{\mu\}$.

Corollary 4.12. *For any symmetric function C, partition λ and positive integer n, it follows from the lemma 4.10 of plethysms properties*

$$0 \,\underline{\otimes}\, \{\lambda\} = \delta_{\lambda,(0)}\{0\} \tag{4.36}$$

$$0 \,\underline{\otimes}\, C = \langle C|\{0\}\rangle\{0\} \tag{4.37}$$

$$\{0\} \,\underline{\otimes}\, \{\lambda\} = \delta_{\ell(\lambda),1}\{0\} = \begin{cases} \{0\} & \textit{if } \ell(\lambda) = 1 \textit{ that is } \{\lambda\} = \{m\} = h_{(m)} \\ 0 & \textit{otherwise}. \end{cases} \tag{4.38}$$

$$\{0\} \,\underline{\otimes}\, C = \sum_{m \geqslant 0} \langle C|h_{(m)}\rangle\{0\} \tag{4.39}$$

$$\begin{aligned} (n\{\lambda\}) \,\underline{\otimes}\, C &= (\{\lambda\} + \cdots + \{\lambda\}) \,\underline{\otimes}\, C \\ &= \sum_{(C)^n} \{\lambda\} \,\underline{\otimes}\, C_{(1)} \cdots \{\lambda\} \,\underline{\otimes}\, C_{(n)} \end{aligned} \tag{4.40}$$

$$\begin{aligned} (-n\{\lambda\}) \,\underline{\otimes}\, C &= (-\{\lambda\} - \cdots - \{\lambda\}) \,\underline{\otimes}\, C \\ &= \sum_{(C)^n} \{\lambda\} \,\underline{\otimes}\, \mathsf{S}(C_{(1)}) \cdots \{\lambda\} \,\underline{\otimes}\, \mathsf{S}(C_{(n)}) \end{aligned} \tag{4.41}$$

where summation over the Sweedler indices is implicitly assumed and where S *is the antipode of the outer Hopf algebra.*

In the next theorem, we use both outer and inner coproducts, distinguished by brackets $(\cdot)$ and $[\cdot]$, respectively.

Theorem 4.13 (main theorem (i))**.** *For any Schur function series Φ and an arbitrary Schur function $\{\pi\}$ the outer coproduct of the plethysm $\{\pi\} \,\underline{\otimes}\, \Phi$ is given by*

$$\Delta(\{\pi\} \,\underline{\otimes}\, \Phi) = \sum_{[\Phi_{(1)}]} \cdots \sum_{[\Phi_{(N^\pi)}]} \sum_{(\Phi)^{N^\pi}} \prod_{(k)=1}^{N^\pi} (\{\pi_{(1)k}\} \,\underline{\otimes}\, \Phi_{(k)[1]}) \otimes (\{\pi_{(2)k}\} \,\underline{\otimes}\, \Phi_{(k)[2]}). \tag{4.42}$$

Proof. The result follows immediately from lemma 4.11 by setting $\lambda = \pi$, replacing μ by Φ and noting the right distributivity of plethysm with respect to addition (4.24). □

Before we draw some conclusions from this theorem, we want to state the even more general result for the coproduct of the plethysm of two Schur function series. Let $\Xi = \sum_\lambda x_\lambda s_\lambda$, with $x_\lambda \in \mathbb{Z}$, be a Schur function series. Such a series is called Schur positive if all the coefficients x_λ are non-negative. A Schur function series is called (strictly) Schur negative if all coefficients are strictly negative. Obviously *any* Schur function series can be decomposed into a Schur positive and a Schur negative part $\Xi = \Xi^+ - \Xi^- = \sum_\lambda x^+_\lambda s_\lambda - \sum_\mu x^-_\mu s_\mu$, with $x^+_\lambda \geqslant 0$ and $x^-_\mu > 0$. In the following, we assume that there are finitely many non-zero terms, in order to restrict the iterated coproducts to finite depth. Due to right distributivity of the plethysm the problem can be reduced to the case $\Xi \underline{\otimes} \{\pi\}$.

Theorem 4.14 (main theorem (ii))**.** *For any Schur function series* $\Xi = \sum_\lambda^{N^+} x^+_\lambda s_\lambda - \sum_\mu^{N^-} x^-_\lambda s_\mu$, *the outer coproduct of* $\Xi \underline{\otimes} \{\pi\}$ *reads*

$$\Delta(\{\Xi\} \underline{\otimes} \{\pi\}) = \sum_{[\pi]} \prod_{k=1}^{N^+} \prod_{l=1}^{x^+_\lambda} \{\lambda_{(1)}\} \underline{\otimes} \{\pi_{(1)(k)(l)[1]}\} \otimes \{\lambda_{(2)}\} \underline{\otimes} \{\pi_{(1)(k)(l)[2]}\}$$
$$\cdot \sum_{[\pi]} \prod_{k=1}^{N^-} \prod_{l=1}^{x^-_\lambda} \{\lambda_{(1)}\} \underline{\otimes} \mathsf{S}(\{\pi_{(2)(k)(l)[1]}\}) \otimes \{\lambda_{(2)}\} \underline{\otimes} \mathsf{S}(\{\pi_{(2)(k)(l)[2]}\}) \qquad (4.43)$$

where the outer product of the two tensor factors has been left unevaluated for clarity. The composite Sweedler index notation indicates three levels of successive outer coproduct and a final inner coproduct (see (4.45)).

Proof. Using the right distributivity (4.24) of the plethysm and corollary 4.12, the sum splits into two parts which can be treated by applying lemma 4.11.

$$\begin{aligned}\Delta(\{\Xi\} \underline{\otimes} \{\pi\}) &= \Delta(\{\Xi^+ - \Xi^-\} \underline{\otimes} \{\pi\}) \\ &= \Delta(\{\Xi^+\} \underline{\otimes} \{\pi_{(1)}\}) \cdot \Delta(\{-\Xi^-\} \underline{\otimes} \{\pi_{(2)}\}) \\ &= \Delta(\{\Xi^+\} \underline{\otimes} \{\pi_{(1)}\}) \cdot \Delta(\{\Xi^-\} \underline{\otimes} \mathsf{S}(\{\pi_{(2)}\})) \qquad (4.44)\end{aligned}$$

Now we consider the first term

$$\begin{aligned}\Delta(\Xi^+ \underline{\otimes} \pi_{(1)}) &= \Delta\left(\sum_{\lambda(k),k=1}^{N^+} x^+_\lambda \{\lambda\} \underline{\otimes} \{\pi_{(1)}\}\right) \\ &= \Delta\left(\prod_{k=1}^{N^+} x^+_\lambda \{\lambda\} \underline{\otimes} \{\pi_{(1)(k)}\}\right) \\ &= \Delta\left(\prod_{k=1}^{N^+} \prod_{l=1}^{x^+_\lambda} \{\lambda\} \underline{\otimes} \{\pi_{(1)(k)(l)}\}\right) \\ &= \sum_{[\pi]} \prod_{k=1}^{N^+} \prod_{l=1}^{x^+_\lambda} (\{\lambda_{(1)}\} \underline{\otimes} \{\pi_{(1)(k)(l)[1]}\}) \otimes (\{\lambda_{(2)}\} \underline{\otimes} \{\pi_{(1)(k)(l)[2]}\}) \qquad (4.45)\end{aligned}$$

and we can expand the second term in an analogous fashion, where the antipode is given by $\mathsf{S}(\{\pi\}) = (-1)^{|\pi|}\{\pi'\}$. Hence, the result follows. □

It is clear by now that the two parts of the main theorem allow the computation of the outer coproduct of the plethysm of a polynomial Schur function series and another Schur function series

$$\Delta(\Xi \underline{\otimes} \Phi). \tag{4.46}$$

The resulting formula is of a remarkable complexity and not displayed. The interested reader should note that these coproducts are of polynomial type, sometimes called Faà di Bruno coproducts. Such coproducts play a crucial role in renormalization theory and are a typical feature of coproducts emerging from composition. For references see [5].

4.3. Application to M_π

While it would be interesting to examine this general case more closely, we want to turn here back to the case of the M_π series treated combinatorially in section 2.3. First, we need a few more facts about the M series itself.

Lemma 4.15. *The inner coproduct of the M series is given by the Cauchy kernel (2.33):*

$$\delta M = M_{[1]} \otimes M_{[2]} = \prod_{i,j}(1 - x_i y_j)^{-1} = \sum_\sigma s_\sigma(x) s_\sigma(y). \tag{4.47}$$

The outer coproduct of the M series is group like:

$$\Delta M = M_{(1)} \otimes M_{(2)} = \prod_i (1 - x_i)^{-1} \prod_j (1 - y_j)^{-1} = M \otimes M \tag{4.48}$$

$$\Delta^{(k-1)} M = M_{(1)} \otimes \cdots \otimes M_{(k)} = M \otimes \cdots \otimes M. \tag{4.49}$$

Proof. These are well-known identities, stated or partly proved in section 2.4, see (2.33) and (2.32). The final formula can either be seen directly from the product form of M or deduced recursively. □

Corollary 4.16 (specialization of main theorem (i)), see proposition 2.3)**.** *For any partition π, the coproduct of $M_\pi = \{\pi\} \underline{\otimes} M \equiv \{\pi\}^{\underline{\otimes} M}$ is given by*

$$\Delta M_\pi = (M_\pi)_{(1)} \otimes (M_\pi)_{(2)} \tag{4.50}$$

$$= (M_\pi \otimes M_\pi) \cdot (\{\pi'_{(1)}\}^{\underline{\otimes} M_{[1]}} \otimes \{\pi'_{(2)}\}^{\underline{\otimes} M_{[2]}}) \tag{4.51}$$

$$= (M_\pi \otimes M_\pi) \cdot \prod_{\xi,\eta<\pi} \prod_{k=1}^{C^\pi_{\xi\eta}} \sum_{\sigma(\xi,\eta,k)} (\{\xi\} \underline{\otimes} \{\sigma(\xi,\eta,k)\}) \otimes (\{\eta\} \underline{\otimes} \{\sigma(\xi,\eta,k)\}) \tag{4.52}$$

where the $C^\pi_{\xi\eta}$ are the Littlewood–Richardson coefficients of outer multiplication. The summations over the $\sigma(\xi,\eta,k)$ are formally over all Schur functions.

Proof. We give an independent proof, since this is the main technical result required to obtain the M_π branchings. First, it should be noted that (4.50) is just Sweedler notation for the decomposition of the coproduct of M_π. Its evaluation serves to define the Sweedler sum $(M_\pi)_{(1)} \otimes (M_\pi)_{(2)}$. Using the suggestive exponential notation, the second form (4.51) comes about by noting that, since $M_{(k)} = M$ for all k, we have

$$\begin{aligned}\Delta M_\pi &= \Delta(\{\pi\}^{\underline{\otimes} M}) = (\Delta(\{\pi\}))^{\underline{\otimes} M}\\ &= (\{\pi\}\otimes\{0\} + \{0\}\otimes\{\pi\} + \Delta'(\{\pi\}))^{\underline{\otimes} M}\\ &= (\{\pi\}\otimes\{0\})^{\underline{\otimes} M}\cdot(\{0\}\otimes\{\pi\})^{\underline{\otimes} M}\cdot\Delta'(\{\pi\}))^{\underline{\otimes} M} \end{aligned}\tag{4.53}$$

where 'proper cuts' (4.7) have been used. The second factor of (4.51) is just $\Delta'(\{\pi\}))^{\underline{\otimes} M}$ expressed in terms of Sweedler sums. To obtain the first factor, it should be noted that as a consequence of (4.27), lemma 4.15 and (4.38), we have

$$(\{\pi\}\otimes\{0\})^{\underline{\otimes} M} = \sum_\sigma \{\pi\}^{\underline{\otimes}\{\sigma\}}\otimes\{0\}^{\underline{\otimes}\{\sigma\}} = \sum_m \{\pi\}^{\underline{\otimes}\{m\}}\otimes\{0\} = \{\pi\}^{\underline{\otimes} M}\otimes\{0\} = M_\pi\otimes\{0\}. \tag{4.54}$$

Similarly $(\{0\}\otimes\{\pi\})^{\underline{\otimes} M} = \{0\}\otimes M_\pi$. Hence,

$$(\{\pi\}\otimes\{0\})^{\underline{\otimes} M}\cdot(\{0\}\otimes\{\pi\})^{\underline{\otimes} M} = M_\pi\otimes\{0\}\cdot\{0\}\otimes M_\pi = M_\pi\otimes M_\pi, \tag{4.55}$$

as required in (4.53) to give the first factor of (4.51). It only remains to calculate the cut coproduct plethysm. This is given by

$$\begin{aligned}\Delta'(\{\pi\}))^{\underline{\otimes} M} &= \left(\sum_{\xi,\eta<\pi} C^\pi_{\xi\eta}\{\xi\}\otimes\{\eta\}\right)^{\underline{\otimes} M} = \prod_{\xi,\eta<\pi}\prod_{k=1}^{C^\pi_{\xi\eta}}(\{\xi\}\otimes\{\eta\})^{\underline{\otimes} M}\\ &= \prod_{\xi,\eta<\pi}\prod_{k=1}^{C^\pi_{\xi\eta}}\sum_{\sigma(\xi,\eta,k)}\{\xi\}^{\underline{\otimes}\sigma(\xi,\eta,k)}\otimes\{\eta\}^{\underline{\otimes}\sigma(\xi,\eta,k)}, \end{aligned}\tag{4.56}$$

precisely as required[11]. □

Reverting to the Schur function notation for plethysms and introducing indeterminates $z = (x, y)$ in connection with the coproducts, it is clear that (4.52) coincides with (2.37). Examples of such coproducts have already been displayed in example 2.4.

The skewing by a series amounts to establishing a branching rule from $\mathsf{GL}(n)$ characters, that is Schur functions, into H_π characters of a subgroup $\mathsf{H}_\pi(n)\subset\mathsf{GL}(n)$ which leaves a tensor of symmetry type π invariant. Such branching rules take the form (2.20), that is $\{\lambda\}\to(\!(\lambda/M_\pi)\!)_\pi$. At this point, the relation between the structure of the Schur function series and the nature of the branching, mentioned in the introductory remarks to this section, can be discerned. While group like series induce branching operators which are algebra homomorphisms, those series M_π with nontrivial plethysms $|\pi|\geqslant 2$ are in general no longer homomorphisms (see section 4.4 and appendix A for further details). The following theorem allows a generalization to arbitrary series M_π.

Theorem 4.17. *For any partitions μ, ν and series M_π one finds the π-skewed branching formula*

$$\begin{aligned}(\{\mu\}\cdot\{\nu\})/M_\pi &= \{\mu/(M_\pi)_{(1)}\}\cdot\{\nu/(M_\pi)_{(2)}\}\\ &= \sum_{\sigma(\xi,\eta,k)}\left\{\mu\Big/\Big(M_\pi\prod_{\xi\eta<\pi}\prod_{k=1}^{C^\pi_{\xi\eta}}\xi\,\underline{\otimes}\,\sigma(\xi,\eta,k)\right\}\cdot\left\{\nu\Big/\Big(M_\pi\prod_{\xi\eta<\pi}\prod_{k=1}^{C^\pi_{\xi\eta}}\eta\,\underline{\otimes}\,\sigma(\xi,\eta,k)\Big)\right\}\end{aligned}\tag{4.57}$$

[11] In (4.60) this expression is denoted by $M_{\pi'_{(1)}}\otimes M_{\pi'_{(2)}}$.

where the summations over the $\sigma(\xi, \eta, k)$ are formally over all Schur functions, and, as usual, $C^{\pi}_{\xi,\eta}$ indicates the number of terms $\{\xi\} \otimes \{\eta\}$ appearing in the coproduct of $\{\pi\}$ for fixed ξ and η.

Proof. We make use of the duality mediated by the Schur scalar product

$$\begin{aligned}\langle (s_\mu \cdot s_\nu)/M_\pi \mid s_\rho\rangle &= \langle s_\mu \cdot s_\nu \mid M_\pi \cdot s_\rho\rangle \\ &= \langle s_\mu \otimes s_\nu \mid \Delta(M_\pi \cdot s_\rho)\rangle \\ &= \langle s_\mu \otimes s_\nu \mid \Delta(M_\pi) \cdot \Delta(s_\rho)\rangle \\ &= \langle s_\mu \otimes s_\nu \mid ((M_\pi)_{(1)} \otimes (M_\pi)_{(2)}) \cdot \Delta(s_\rho)\rangle \\ &= \langle s_\mu/(M_\pi)_{(1)} \otimes s_\nu/(M_\pi)_{(2)} \mid \Delta(s_\rho)\rangle \\ &= \langle s_\mu/(M_\pi)_{(1)} \cdot s_\nu/(M_\pi)_{(2)} \mid s_\rho\rangle. \end{aligned} \tag{4.58}$$

Since the Schur scalar product is nondegenerate, and since s_ρ was arbitrary and the Schur functions form a basis the result follows, with the final step made by identifying (4.50) with (4.52). This proof does not make use of the property that M_π is group like and thus applies quite generally for all π. □

Before we can prove the π-generalized version of the Newell–Littlewood theorem we need a lemma about the coproduct of inverse M_π series.

Lemma 4.18. *Every M_π series has as inverse the L_π series; furthermore, the part obtained by proper cuts of the coproduct is invertible too*

$$M_\pi \cdot L_\pi = 1 \tag{4.59}$$

$$M_{\pi'_{(1)}} L_{\pi'_{(1)}} \otimes M_{\pi'_{(2)}} L_{\pi'_{(2)}} = 1 \otimes 1. \tag{4.60}$$

Proof. From the observation that $M \cdot L = 1$, we obtain

$$M_\pi \cdot L_\pi = (\{\pi\} \underline{\otimes} M) \cdot (\{\pi\} \underline{\otimes} L) = \{\pi\} \underline{\otimes} (M \cdot L) = \{\pi\} \underline{\otimes} \{0\} = \{0\} = 1. \tag{4.61}$$

An alternative way to see this reads

$$\begin{aligned} M_\pi \cdot L_\pi &= \{\pi\} \underline{\otimes} M \cdot (-\{\pi\}) \underline{\otimes} M = \sum_{(M)} \{\pi\} \underline{\otimes} M_{(1)} \cdot (-\{\pi\}) \underline{\otimes} M_{(2)} \\ &= (\{\pi\} - \{\pi\}) \underline{\otimes} M = 0 \underline{\otimes} M = \delta_{\{0\},M} = 1 \end{aligned} \tag{4.62}$$

showing that using the identities of the algebra of plethysm, equations (4.25) and (4.37), is much weaker than the first consideration, since we had to make use of the fact that M is group like. In particular this shows that the plethysm by a negative Schur function $-\{\pi\}$ applied to a series Φ gives the inverse $(\Phi_\pi)^{-1}$ if and only if Φ is group like. The previous result can be used to establish the following identity:

$$\begin{aligned} 1 \otimes 1 = \{0\} \otimes \{0\} = \Delta(\{0\}) &= \Delta(M_\pi \cdot L_\pi) = \Delta(M_\pi) \cdot \Delta(L_\pi) \\ &= \left(M_{\pi_{(1)}} \otimes M_{\pi_{(2)}}\right) \cdot \left(L_{\pi_{(1)}} \otimes L_{\pi_{(2)}}\right) \\ &= \left(M_{\pi_{(1)}} \cdot L_{\pi_{(1)}}\right) \otimes \left(M_{\pi_{(2)}} \cdot L_{\pi_{(2)}}\right) \\ &= \left(M_\pi M_{\pi'_{(1)}} L_\pi L_{\pi'_{(1)}}\right) \otimes \left(M_\pi M_{\pi'_{(2)}} L_\pi L_{\pi'_{(2)}}\right) \\ &= \left(M_\pi L_\pi M_{\pi'_{(1)}} L_{\pi'_{(1)}}\right) \otimes \left(M_\pi L_\pi M_{\pi'_{(2)}} L_{\pi'_{(2)}}\right) \\ &= \left(M_{\pi'_{(1)}} L_{\pi'_{(1)}}\right) \otimes \left(M_{\pi'_{(2)}} L_{\pi'_{(2)}}\right) \end{aligned} \tag{4.63}$$

where one should remember that primes at Sweedler indices denote the proper cuts of the coproduct. □

Products of characters of $\mathsf{H}_\pi(n) \subset \mathsf{GL}(n)$ can be obtained using the following generalization of the Newell–Littlewood theorem.

Theorem 4.19 (π-Newell–Littlewood theorem, proposition 2.7)**.** *Let $(\!(\mu)\!)_\pi$, $(\!(\nu)\!)_\pi$ be formal universal characters of $\mathsf{H}_\pi(n)$ defined in terms of Schur functions by $(\!(\mu)\!)_\pi = \{\mu/L_\pi\}$ and $(\!(\nu)\!)_\pi = \{\nu/L_\pi\}$, as in (2.25). Then,*

$$(\!(\mu)\!)_\pi \cdot (\!(\nu)\!)_\pi = (\!(\mu/(\{\pi'_{(1)}\}^{\underline{\otimes} M_{[1]}}) \cdot \nu/(\{\pi'_{(2)}\}^{\underline{\otimes} M_{[2]}}))\!)_\pi$$
$$= \sum_{\sigma(\xi,\eta,k)} \left(\!\!\left(\left\{\mu/(\prod_{\xi\eta<\pi}\prod_{k=1}^{C^\pi_{\xi\eta}}\{\xi\}\,\underline{\otimes}\,\sigma(\xi,\eta,k)\right\} \cdot \left\{\nu/(\prod_{\xi\eta<\pi}\prod_{k=1}^{C^\pi_{\xi\eta}}\{\eta\}\,\underline{\otimes}\,\sigma(\xi,\eta,k)\right\}\right)\!\!\right)_\pi \tag{4.64}$$

where $\sigma(\xi,\eta,k)$ that are associated with the Sweedler indices are formally summed over all Schur functions.

Proof. We make use of duality and of the second identity of lemma 4.18 to calculate the product of $\mathsf{H}_\pi(n)$ characters directly in terms of Schur functions:

$$\begin{aligned}\langle(\!(\mu)\!)_\pi \cdot (\!(\nu)\!)_\pi \mid s_\rho\rangle &= \langle\mu\otimes\nu \mid L_\pi\otimes L_\pi \cdot \Delta s_\rho\rangle\\
&= \left\langle\mu\otimes\nu \,\middle|\, \left(M_{\pi'_{(1)}}L_{\pi'_{(1)}}\right)\otimes\left(M_{\pi'_{(2)}}L_{\pi'_{(2)}}\right)L_\pi\otimes L_\pi\cdot\Delta s_\rho\right\rangle\\
&= \left\langle\mu\otimes\nu \,\middle|\, M_{\pi'_{(1)}}\otimes M_{\pi'_{(2)}}\cdot L_{\pi'_{(1)}}\otimes L_{\pi'_{(2)}}\cdot L_\pi\otimes L_\pi\cdot\Delta s_\rho\right\rangle\\
&= \left\langle\mu\big/M_{\pi'_{(1)}}\otimes\nu\big/M_{\pi'_{(2)}} \,\middle|\, \Delta L_\pi\cdot\Delta s_\rho\right\rangle\\
&= \left\langle\mu\big/M_{\pi'_{(1)}}\otimes\nu\big/M_{\pi'_{(2)}} \,\middle|\, \Delta(L_\pi\cdot s_\rho)\right\rangle\\
&= \left\langle\left(\!\left(\mu\big/M_{\pi'_{(1)}}\cdot\nu\big/M_{\pi'_{(2)}}\right)\!\right)_\pi \,\middle|\, s_\rho\right\rangle.\end{aligned} \tag{4.65}$$

The conclusion follows from nondegeneracy of the Schur scalar product and completeness of the Schur basis. □

Proof. Alternatively, and more directly, by virtue of the $\mathsf{GL}(n) \supset \mathsf{H}_\pi(n)$ branching rule (2.20) and its inverse (2.25), the multiplicity of the $\mathsf{H}_\pi(n)$ character $(\!(\rho)\!)_\pi$ in the product $(\!(\mu)\!)_\pi \cdot (\!(\nu)\!)_\pi$ is given in terms of characters of $\mathsf{GL}(n)$, that is to say Schur functions, by

$$\begin{aligned}\langle(\!(\mu)\!)_\pi\cdot(\!(\nu)\!)_\pi \mid (\!(\rho)\!)_\pi\rangle &= \langle\{(\mu/L_\pi\cdot\nu/L_\pi)/M_\pi\} \mid \{\rho/L_\pi\}\rangle\\
&= \langle\{\mu/L_\pi\cdot\nu/L_\pi\} \mid M_\pi\cdot\{\rho/L_\pi\}\rangle\\
&= \langle\{\mu/L_\pi\}\otimes\{\nu/L_\pi\} \mid \Delta(M_\pi\cdot\{\rho/L_\pi\})\rangle\\
&= \langle\{\mu/L_\pi\}\otimes\{\nu/L_\pi\} \mid \Delta(M_\pi)\cdot\Delta(\{\rho/L_\pi\})\rangle\\
&= \langle\{\mu/L_\pi\}\otimes\{\nu/L_\pi\} \mid (M_\pi\otimes M_\pi)\\
&\quad\cdot(\{\pi'_{(1)}\}^{\underline{\otimes} M_{[1]}}\otimes\{\pi'_{(2)}\}^{\underline{\otimes} M_{[2]}})\cdot\Delta(\{\rho/L_\pi\})\rangle\\
&= \langle\{\mu/(L_\pi M_\pi(\{\pi'_{(1)}\}^{\underline{\otimes} M_{[1]}}))\}\\
&\quad\otimes\{\nu/L_\pi M_\pi(\{\pi'_{(2)}\}^{\underline{\otimes} M_{[2]}}))\} \mid \Delta(\{\rho/L_\pi\})\rangle\\
&= \langle\{\mu/(\{\pi'_{(1)}\}^{\underline{\otimes} M_{[1]}})\}\cdot\{\nu/(\{\pi'_{(2)}\}^{\underline{\otimes} M_{[2]}})\} \mid \{\rho/L_\pi\}\rangle.\end{aligned} \tag{4.66}$$

Here (as in the first proof above) use has been made in particular of the proper cut coproduct plethysm (4.51), evaluated by means of (4.52). However, note that although the inverse series (equation (4.59)) are required, the cut product inverse series are not invoked (equation (4.60)). Again the nondegeneracy of the Schur scalar product and the completeness of the Schur basis allow us immediately to draw the conclusion (4.64), which is in accord with proposition 2.7. This proof is a direct application of definition (4.17) for $\Phi = M_\pi$ and $\Phi^{-1} = L_\pi$. □

4.4. Scalar products, Cauchy kernels and plethystic generalizations

In the present section, we discuss briefly the role of the Cauchy kernel (2.33) and its inverse, the Cauchy–Binet formula and their generalizations in the branching process. In appendix A, we give an additional description in terms of tangles which we consider to be helpful in contemplating this structure.

The coproducts of group like branchings induce algebra homomorphisms, which are nontrivial in the sense that they still induce new representations, see the V series in table 1, but lie in the trivial cohomology class in the sense of algebra deformation theory.

$$(\Delta\Phi)(x, y) = \Phi(x)\Phi(y) \tag{4.67}$$

$$\begin{aligned} m_\phi(x \otimes y) &= (\partial\phi)(x_{(1)}, y_{(1)})x_{(2)} \cdot y_{(2)} \\ &= \Phi^{-1}(\Phi(x) \cdot \Phi(y)). \end{aligned} \tag{4.68}$$

The relation between the Schur function series and the linear forms used in branching operators (4.13) is given via the Schur scalar product $\phi = (\Phi|.)$. However, if $\mathsf{O}(n)$ or $\mathsf{Sp}(n)$ are considered, one is confronted with branchings which are mediated by branching operators which fail to be group like. From the above discussion, we know how the coproduct of series is computed and that the Cauchy kernel plays a key role in the derivation of these results. Moreover, due to plethysms, convolution products of the Cauchy kernel appear, where in one slot or both slots a plethysm takes place.

While the Cauchy kernel (2.33) presents a $0 \to 2$ map, producing from the 'zero series' 1 a second rank tensor, the Schur scalar product is just a $2 \to 0$ map which takes a second rank tensor to a number.

$$1 \to \prod_{i,j}(1 - x_i y_j)^{-1} = \sum_\sigma u_\sigma \otimes v_\sigma \tag{4.69}$$

$$x \otimes y \to (x|y). \tag{4.70}$$

Note that u_σ, v_σ is a dual pair of bases with respect to the Schur scalar product. Hence, the scalar product and Cauchy kernel are dual objects, this is graphically displayed in equation (A.5). This opens the possibility of studying scalar products which are dual to the coevaluations used in the coproduct formulae of Schur function series. Such a view amounts to introducing a deformation in the product, not in the coproduct. Moreover, this duality explains why the Cauchy kernel

$$\mathsf{C}(x, y) := \prod_{i,j}(1 - x_i y_j)^{-1} = \sum_\sigma u_\sigma \otimes v_\sigma \tag{4.71}$$

is a reproducing kernel for Schur function (series)—for, if F is an arbitrary symmetric function, we have

$$\mathsf{C}\cdot F(x) := \sum_{\sigma} u_\sigma(x)\langle v_\sigma|F\rangle \equiv F(x). \tag{4.72}$$

Displayed in the graphical notation this property resembles a Reidemeister move[12].

Before we discuss scalar products we consider coproducts with primitive elements, that is power sum symmetric functions. The outer coproduct of a one part power symmetric function p_n has exactly two parts

$$\Delta(p_n) = p_n \otimes 1 + 1 \otimes p_n. \tag{4.73}$$

From our main theorem (i) (theorem 4.13) we obtain the fact that any power sum plethysm by a group like series G remains group like,

$$\begin{aligned}\Delta(\{p_n\}\underline{\otimes} G) &= \Delta(p_n)\underline{\otimes} G = (p_n \otimes p_0 + p_0 \otimes p_n)\underline{\otimes} G\\ &= (p_n \otimes p_0)\underline{\otimes} G \cdot (p_0 \otimes p_n)\underline{\otimes} G\\ &= \{p_n\}\underline{\otimes} G \otimes \{p_n\}\underline{\otimes} G,\end{aligned} \tag{4.74}$$

where in the last line only the first Sweedler term in the inner coproduct $\delta(\{\lambda\}) = \{\lambda\}\otimes\{|\lambda|\}+\cdots$ survives. However, due to the inner coproduct this fails for plethysms of the form $\{p_\lambda\}\underline{\otimes} G$ for λ a partition having more than one part.

The interesting terms which occur for non-group like coproducts induce a twisting exactly such that the lack of being a homomorphism is compensated. This can be seen explicitly in proposition 2.3, where the right-hand side has two terms $M_\pi(x)M_\pi(y)$ resembling the part which would be group like, and the twist $\prod\prod\sum s_{\xi\underline{\otimes}\sigma}(x)s_{\eta\underline{\otimes}\sigma}(y)$, related to the proper cuts of $\Delta(\pi)$. Due to the general theory of such deformations and the fact that we are dealing with group characters, we can conclude that this twisting is a 2-cycle, its dual is also a 2-cocycle. Furthermore, we note that the M series is the unit of the inner product and that the Cauchy kernel can be obtained by applying the inner coproduct to the M series, $\delta M(x, y) = \mathsf{C}(x, y)$. This is the way the Cauchy kernel entered in the branching formulae derived above.

Not all plethysms give rise to different 2-cycles. For example, since the proper cut part of the coproduct of $\{2\}$ and $\{1^2\}$ are identical, they induce the same 2-cycle, that is the Cauchy kernel. This shows that there are homologous 2-cycles, and a major question is to classify all such cycles.

Switching to the dual setting, we consider $2 \to 0$ maps, which generalize the Schur scalar product. For example, for the plethystic deformation using $\{1^3\}$, we obtain a new scalar product

$$(x \mid y)_{\{1^3\}} = \langle x_{(1)} \mid \{1^2\}\underline{\otimes}\, y_{(1)}\rangle\langle\{1^2\}\underline{\otimes}\, x_{(2)} \mid y_{(2)}\rangle \tag{4.75}$$

while the Schur scalar product is the special case

$$\langle x \mid y\rangle = (x \mid y)_{\{2\}} = (x \mid y)_{\{1^2\}}. \tag{4.76}$$

The trivial case $(x \mid y)_{\{1\}} = \delta_{x,\{0\}}\delta_{y,\{0\}}$ belongs to the group like case. Dualizing the twist for a general π introduced in proposition 2.3, we obtain the family of scalar products

$$(x \mid y)_\pi = \prod_{\xi,\eta}\prod^{C^\pi}\langle\{\xi\}\underline{\otimes}\, x \mid \{\eta\}\underline{\otimes}\, y\rangle \tag{4.77}$$

[12] In fact, the variable y can be considered here to be a linear form on x and the action written '$\sum_{(y)}\mathsf{C}(x, y)F(y)$'.

which in general is easily seen to be nonisomorphic and noncohomologous to the Schur scalar product. All these scalar products can be used to introduce twists and all of these scalar products induce associative product deformations, since we are dealing with group characters, hence are 2-cocycles.

Our brief discussion shows that the new branchings also open a new research area in deformation theory, since we have a hand on the mechanism to introduce families of appropriate 2-cocycles. A major problem is to classify these 2-cocycles and to provide tools to be able to decide if two such 2-cocycles are cohomologous. Moreover, the identification of the groups related to these 2-cocycles is important, in relation to the question of whether these families exhaust the space of cohomology classes of 2-cocycles. Some further discussion has been put into appendix A.

5. Conclusions

The present work opens the door into a fascinating new field of group representation techniques. Starting with some Hopf algebraically motivated questions posed in [11] it was possible to derive group branchings tied to the plethystic series of M_π and L_π types. Branchings based on these series led by direct calculations to SL groups considered as subgroups of appropriate GL groups, $\mathsf{SL}(n) \equiv \mathsf{H}_{\{1^n\}} \subset \mathsf{GL}(n)$. While the general theory works with formal characters in the inductive limit, finite representation spaces require modification rules to cope with syzygies. This is beyond the scope of the present work, but hints were given in the generalization examples. These examples uncovered non-semisimple and non-reductive groups, such as affine groups, and more general semi-direct product groups. A case study was presented for $\mathsf{H}_{1^3}(3) \equiv \mathsf{SL}(3)$ and $\mathsf{H}_{1^3}(4) \supset \mathsf{GL}(3) \times \mathsf{GL}(1)$. Character tables, modification rules and products of characters were derived.

We paused the combinatorial exploration to introduce the Hopf algebra machinery which proved to be a powerful way of encoding the complexity of the new branchings. To our knowledge, these branchings have not appeared in the literature before, since a purely combinatorial route to them would be rather difficult to discern. However, using Hopf algebraic branching operators, we can tie to any Schur function series a branching process. If we restrict ourselves to M_π series, these are matrix subgroups of $\mathsf{GL}(n)$ which fix a tensor of Young symmetry T^π. Particular cases are the branchings $\mathsf{GL}(n) \downarrow \mathsf{GL}(n-1)$, $\mathsf{GL}(n) \downarrow \mathsf{O}(n)$ and $\mathsf{GL}(2n) \downarrow \mathsf{Sp}(2n)$ fixing a vector v_i, a symmetric tensor $g_{ij} = g_{ji}$ and an antisymmetric tensor $f_{ij} = -f_{ji}$. The SL groups appear through the same mechanism as $\mathsf{H}_{1^n}(n) \subset \mathsf{GL}(n)$ fixing a volume form (antisymmetric highest rank tensor in n dimension).

The main theorem parts (i) and (ii) (theorems 4.13 and 4.14) allow in principle the choice of a linear combination of tensors of different Young symmetry types. The formulae obtained have a close relation to vertex operators, which are composed of two Schur function series $\Gamma(z) = H(z)E^\perp(z^{-1})$, see [32]. This supports our finding that affine groups can occur. The question about canonical forms of tensors of a certain Young type arises and of their physical relevance.

Related to this is the question of whether Hall–Littlewood symmetric functions can be related to the plethystic scalar products, which we defined in section 4.4. Starting from a nontrivial group like branching, one obtains scalar products which have a kernel establishing a residual symmetry in the Schur function series, see the V series in table 1. It is known that the freedom of choosing q to be a root of unity is related to representation theory over finite geometries. This observation should be contrasted with the arbitrary introduction of q-generalizations via a braided grade group, which has no *a priori* geometrical meaning. The Hopf algebraic treatment may also open a way to understand the modification rules of the involved groups in a systematic way, so that specific case can be examined in the light of more general results.

Finally, the Hopf algebraic treatment does not make explicit use of the ground field. Hence, our methods are in principle applicable to G-sets, working in the Burnside ring, and representations of pro-finite groups on them. This leads to the realm of modular representation theory and should provide deep insights and beautiful combinatorics.

From a physical point of view, we want to emphasize that groups fixing higher rank tensors are related to nonlinear models. For example, the antisymmetric tensor ϵ_{abcd} in n dimensions is saturated by four 'fields' ψ^a. The related invariants are no longer of binary type and hence necessarily nonlinear. Such tensors provide 'interaction terms' like $\epsilon_{abcd}\psi^b\psi^c\psi^d$ in the field equations. There is thus a host of possible applications for our methods, which we hope to explore in future work.

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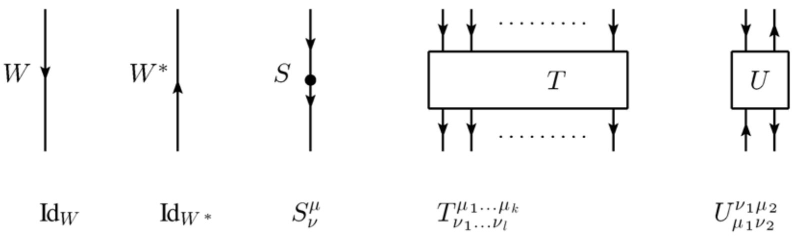
Appendix A. Graphical calculus for plethystic (co)scalar products

In this appendix, we provide enough notion of graphical calculus [23, 24, 30, 31, 41] to be able to discuss the structure of the 2-cocycles and coscalar products involved in the branching process of the plethystic branchings. The method consist in a graphical representation of the index structure of tensors. A detailed exposition may be found in [10].

Let W be a (finite dimensional) complex vector space and W^* its dual. Choose two (nonintersecting) horizontal lines, an input line and an output line, where two sets of a non-negative number of vertices are positioned. We depict W by a line with downward orientation[13] connecting an upper and lower vertex point and depict W^* by a similar line with upward orientation:



(A.1)

$$\mathrm{Id}_W \qquad \mathrm{Id}_{W^*} \qquad S^{\mu}_{\nu} \qquad T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l} \qquad U^{\nu_1\mu_2}_{\mu_1\nu_2} \tag{A.2}$$

and all summations of these tensors are over the ν indices (inputs). Operations of maps are represented as dots or boxes in these tangles[14]. The number of input and output lines of a map

[13] This is the 'pessimistic arrow of time', as coined by Z Oziewicz in his talk at the ICCA6, 1999, in Ixtapa.

[14] One may read these diagrams as a type of flow chart, if the mathematical background of monoidal tensor categories is neglected.

may differ, one speaks then of a $k \to l$ map. Products and coproducts are such special maps of valence $2 \to 1$ and $1 \to 2$

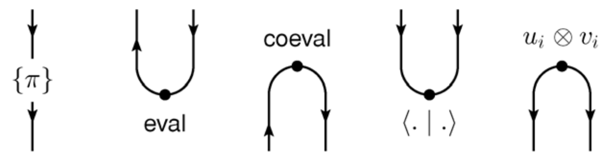
(A.3)

where we have omitted the orientation. In the case of symmetric functions, the product is the outer product of symmetric functions and the coproduct is the outer coproduct of symmetric functions. The space $W = \oplus V^{\otimes^n}$ is the graded space of symmetric functions in infinitely many variables and V is the grade one space generated by the variables x_i.

We use the *same graphical notation* for series of Schur functions too. Hence, we depict the M series or an $A, B, C, D, \ldots, X, \ldots$ series also by a single line. This can be done due to the fundamental theorem of symmetric functions [22], which allows us to regard bases, such as the elementary symmetric functions, complete symmetric functions, Schur functions etc, as new generators of the graded space W. The product of two M series is hence given as

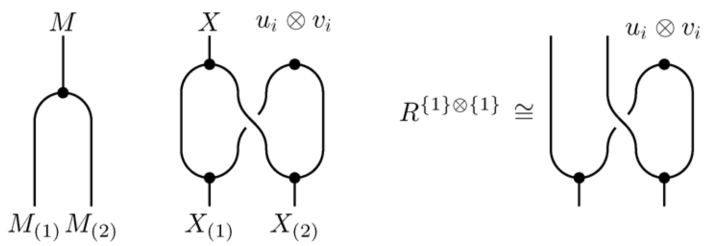
$$m(M(x) \otimes M(y)) \cong M(x, y)|_{x=y} = M(x, x)$$
$$\sum_m \{m\} \cdot \sum_n \{n\} = \sum_{n,m} \{m\} \cdot \{n\} = \sum_{n,m} \sum_\pi C^\pi_{m,n} \{\pi\} \tag{A.4}$$

which is a new series but may not have a standard name like $A, B, C, \ldots$. The same holds true for coproducts. Before we consider such coproducts, we introduce a graphical symbol for the plethysm operation on a Schur function $W[s_\pi] \equiv \{\pi\} \underline{\otimes} W$. Furthermore, we need the tangles for evaluation, coevaluation, scalar products and coscalar products, i.e., the tangle for the Cauchy kernel $u_i \otimes v_i$ where u_i and v_i are mutually dual bases $\langle u_i | v_j \rangle = \delta_{i,j}$. We depict all this as



(A.5)

Now we start to depict the coproduct for the M series (or any group like series $Y \in \{L, M, V, \ldots\}$) and that of a series $X \in \{A, B, C, D, \ldots\}$.



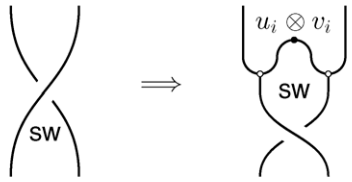
(A.6)

This shows that the deformation for X like series comes up with a deformation which is built from the Cauchy kernel. Hence, the coproduct of X like series can be understood as induced by a coquasitriangular structure:

$$\mathsf{sw} \circ \Delta_X = R^{X'_{(1)} \otimes X'_{(2)}} \circ \Delta, \qquad \forall X \in \{A, B, C, D, \ldots\}. \tag{A.7}$$

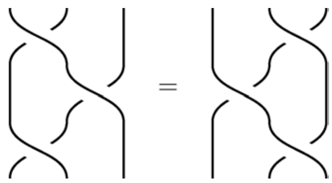
For analogous results in the antisymmetric case see [1] where it was shown that $\mathsf{sw} \circ R$ is actually a universal R-matrix and that it fulfils coquasitriangularity. The preceding equation (A.7) resembles the standard definition of a universal R-matrix, see [15, 33].

To clarify this point, note that the cap tangle in (A.5), i.e. the Cauchy kernel $\sum_i u_i \otimes v_i$, can be used to deform the twist map sw as shown in the right-hand side of equation (A.6). The Cauchy kernel constitutes a quasitriangular structure; this needs a separate proof in the involved case displayed below in equation (A.12). The crossing tangle changes under the twist into the form displayed below using associativity and commutativity:



(A.8)

The left tangle depicts the switch sw, while the right tangle defines the deformed switch $\mathsf{sw} \circ R^{u_i \otimes v_i}$. Due to the cocycle condition this amounts saying that the deformed switch also satisfies the braid equation, sometimes called quantum Yang–Baxter relation,



(A.9)

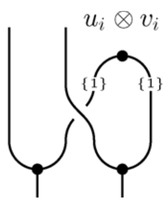
where the crossings are either switches sw (trivial solution) or the deformed crossing $\mathsf{sw} \circ R^{u_i \otimes v_i}$ of equation (A.8).

The elements X are all given by plethysms of the form $A = \{1^2\} \underline{\otimes} M$, $B = \{1^2\} \underline{\otimes} L$, $C = \{2\} \underline{\otimes} L$, $D = \{2\} \underline{\otimes} M$, in general by a plethysm with $\{1^2\}$ or $\{2\}$ of a group like series Y. That these two plethysms act in the same way stems from the fact that they possess the same proper cut coproduct

$$\Delta'\{2\} = \Delta'\{1^2\} = \{1\} \otimes \{1\}. \tag{A.10}$$

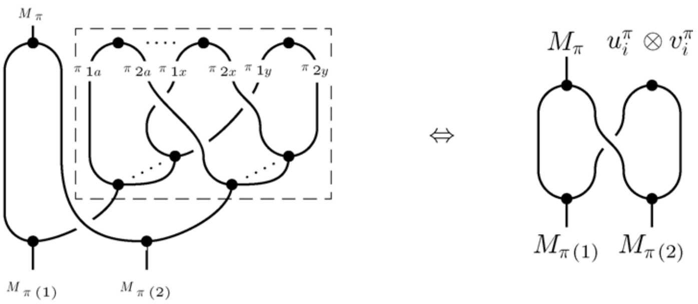
The R-matrix related to these coproducts is given by the plethysms of a sufficient number of Cauchy kernels (here one) with the proper cut coproduct of the involved Schur function $\{\pi\}$. This reads as tangle

$$R^{\Delta'\{2\}} = R^{\Delta'\{1^2\}} = R^{\{1\}\otimes\{1\}} \cong$$



(A.11)

Since the deformed coproducts are derived from group characters we know that these coproducts are coassociative. This allows one to conclude that the involved 2-chains are actually 2-cycles. The trivial 2-cycle $\eta \otimes \eta$ yields a trivial deformation $R = 1 \otimes 1$. The next nontrivial case is given by $R^{\{1\}\otimes\{1\}} = u_i \otimes v_i$, i.e. induced by the Cauchy kernel itself. The new branchings induced by plethysms of weight greater than 2 are then obtained by applying the proper cut coproduct elements as plethysms to a sufficient number of coevaluations (Cauchy kernels). The resulting formula is that of proposition 2.3, resp. corollary 4.16. If the proper cut coproduct terms of π are indexed by $a\ldots xy$, this deformation reads as a tangle



(A.12)

which might be called 'weeping willow' diagram. The framed part of this tangle is again a $0 \to 2$ tangle, hence a 2-cycle or coscalar product is denoted as $u_i^\pi \otimes v_i^\pi$. These 2-cycles are in general homologically different; however, there are degeneracies as $(u)^{\Delta'\{2\}} = u_i^{\{1\}} \otimes v_i^{\{1\}} = (u)^{\Delta'\{1^2\}}$ clearly shows. Of course, all of the above arguments dualize to yield a theory of 2-cocycles and deformed products. A major question which has to be solved is to classify these scalar products.

Appendix B. Combinatorial proofs of propositions 2.5 and 2.7

To prove in a combinatorial way proposition 2.5 we require, in addition to our previous combinatorial development, hence using no Hopf formalism, a pair of lemmas.

Lemma B.20. *For any partitions* λ, μ *and* ν

$$(\{\mu\}\{\nu\})/\{\lambda\} = \sum_{\sigma,\tau} C^\lambda_{\sigma\tau}\{\mu/\sigma\}\{\nu/\tau\}. \tag{B.1}$$

Proof. In order to prove this, we must be more precise about what the left-hand side really means. It can be written more explicitly in the form

$$(\{\mu\}\{\nu\})/\{\lambda\} = (s_\mu(x)s_\nu(x))/s_\lambda(x) = \sum_\rho C^\rho_{\mu\nu} s_{\rho/\lambda}(x). \tag{B.2}$$

Now consider the following two expansions of the same Schur function products:

$$\begin{aligned} s_\mu(x,y)s_\nu(x,y) &= \sum_\rho C^\rho_{\mu\nu} s_\rho(x,y) = \sum_{\rho,\lambda} C^\rho_{\mu\nu} s_{\lambda/\rho}(x) s_\lambda(y) \\ &= \sum_\lambda s_\lambda(y)\left(\sum_\rho C^\rho_{\mu\nu} s_{\lambda/\rho}(x)\right), \end{aligned} \tag{B.3}$$

and

$$\begin{aligned} s_\mu(x,y)s_\nu(x,y) &= \sum_{\sigma,\tau} s_{\mu/\sigma}(x)s_\sigma(y)s_{\nu/\tau}(x)s_\tau(y) \\ &= \sum_{\sigma,\tau} C^\lambda_{\sigma\tau} s_{\mu/\sigma}(x)s_{\nu/\tau}(x)s_\lambda(y) \\ &= \sum_\lambda s_\lambda(y)\left(\sum_{\sigma,\tau} C^\lambda_{\sigma\tau} s_{\mu/\sigma}(x)s_{\nu/\tau}(x)\right). \end{aligned} \tag{B.4}$$

Comparing these two expansions we see that

$$\sum_{\rho} C^{\rho}_{\mu\nu} s_{\lambda/\rho}(x) = \sum_{\sigma,\tau} C^{\lambda}_{\sigma\tau} s_{\mu/\sigma}(x) s_{\nu/\tau}(x). \tag{B.5}$$

Thanks to (B.2) this is all that is required to prove lemma B.20. □

Our second lemma is a linear extension of this.

Lemma B.21. *Let $Z(x)$ be any series of Schur functions*

$$Z(x) = \sum_{\lambda} z_\lambda s_\lambda(x), \tag{B.6}$$

and let the coproduct of Z take the form

$$Z(x, y) = \sum_{\sigma,\tau} C^{Z}_{\sigma\tau} s_\sigma(x) s_\tau(y). \tag{B.7}$$

Then

$$(\{\mu\}\{\nu\})/Z = \sum_{\sigma,\tau} C^{Z}_{\sigma\tau} \{\mu/\sigma\}\{\nu/\tau\}. \tag{B.8}$$

Proof. First note that

$$Z(x, y) = \sum_{\lambda} z_\lambda s_\lambda(x, y) = \sum_{\lambda,\sigma,\tau} z_\lambda C^{\lambda}_{\sigma\tau} s_\sigma(x) s_\tau(y), \tag{B.9}$$

so that

$$C^{Z}_{\sigma\tau} = \sum_{\lambda} z_\lambda C^{\lambda}_{\sigma\tau}. \tag{B.10}$$

Then, using lemma B.20

$$\begin{aligned}(\{\mu\}\{\nu\})/Z &= \sum_{\lambda} z_\lambda (s_\mu(x) s_\nu(x))/s_\lambda(x) \\ &= \sum_{\lambda} z_\lambda \sum_{\sigma\tau} C^{\lambda}_{\sigma\tau} s_{\mu/\sigma}(x) s_{\nu/\tau}(x) \\ &= \sum_{\sigma,\tau} C^{Z}_{\sigma\tau} \{\mu/\sigma\}\{\nu/\tau\},\end{aligned} \tag{B.11}$$

as required. □

Proof of proposition 2.5. Returning to proposition 2.5 its proof amounts to replace Z by M_π in lemma B.21 and substituting into it the $M_\pi(x, y)$ decomposition from proposition 2.3. This gives

$$\begin{aligned}M_\pi(x, y) &= \sum_{\sigma,\tau} C^{M_\pi}_{\sigma\tau} s_\sigma(x) s_\tau(y) \\ &= M_\pi(x) M_\pi(y) \prod_{0\neq\xi,\eta\neq\pi} \prod_{k=1}^{C^{\pi}_{\xi\eta}} \sum_{\sigma(\xi,\eta,k)} s_{\xi\otimes\sigma(\xi,\eta,k)}(x) s_{\eta\otimes\sigma(\xi,\sigma,k)}(y).\end{aligned} \tag{B.12}$$

Using this in (B.8) with $Z = M_\pi$ then gives (2.46) as required to prove proposition 2.5. □

Proof of proposition 2.7. To evaluate the required product of formal characters of $\mathsf{H}_\pi(n)$, one first expresses them in terms of characters of $\mathsf{GL}(n)$ using (2.25), carries out the product in $\mathsf{GL}(n)$ and then expresses the resulting Schur functions back in terms of formal characters of $\mathsf{H}_\pi(n)$ using the branching rule (2.20). This procedure gives

$$(\!(\mu)\!)(\!(\nu)\!) = \{\mu/M_\pi^{-1}\}\{\nu/M_\pi^{-1}\} = (\!((\{\mu/M_\pi^{-1}\}\{\nu/M_\pi^{-1}\})/M_\pi)\!) \tag{B.13}$$

$$= \sum_{\sigma(\xi,\eta,k)} \left(\!\!\left(\left\{\mu \Big/ (M_\pi^{-1} M_\pi \prod_{\substack{0\neq\xi\\ \eta\neq\pi}} \prod_{k=1}^{C^\pi_{\xi\eta}} \xi \underline{\otimes} \sigma(\xi,\eta,k)\right\} \cdot \left\{\nu \Big/ (M_\pi^{-1} M_\pi \prod_{\substack{0\neq\xi\\ \eta\neq\pi}} \prod_{k=1}^{C^\pi_{\xi\eta}} \eta \underline{\otimes} \sigma(\xi,\eta,k)\right\}\right)\!\!\right) \tag{B.14}$$

$$= \sum_{\sigma(\xi,\eta,k)} \left(\!\!\left(\left\{\mu \Big/ \prod_{\substack{0\neq\xi\\ \eta\neq\pi}} \prod_{k=1}^{C^\pi_{\xi\eta}} \xi \underline{\otimes} \sigma(\xi,\eta,k)\right\} \left\{\nu \Big/ \prod_{\substack{0\neq\xi\\ \eta\neq\pi}} \prod_{k=1}^{C^\pi_{\xi\eta}} \eta \underline{\otimes} \sigma(\xi,\eta,k)\right\}\right)\!\!\right), \tag{B.15}$$

which completes the proof. □

Appendix C. Tables

C.1. $\mathsf{H}_3(4)$ *formal characters*

Dimensions of $\mathsf{GL}(4) \downarrow \mathsf{H}_3(4)$ *irreps:*

$\{\lambda\}_{\dim}$	$(\!(\lambda/M_3)\!)_{\dim}$	$\{\lambda\}_{\dim}$	$(\!(\lambda/M_3)\!)_{\dim}$
$\{0\}_1$	$(\!(0)\!)_1$	$\{1\}_4$	$(\!(1)\!)_4$
$\{11\}_6$	$(\!(11)\!)_6$	$\{111\}_4$	$(\!(111)\!)_4$
$\{1^4\}_1$	$(\!(1^4)\!)_1$	$\{1^5\}_0$	$(\!(1^5)\!)_0$
$\{1^6\}_0$	$(\!(1^6)\!)_0$	$\{2\}_{10}$	$(\!(2)\!)_{10}$
$\{21\}_{20}$	$(\!(21)\!)_{20}$	$\{211\}_{15}$	$(\!(211)\!)_{15}$
$\{2111\}_4$	$(\!(2111)\!)_4$	$\{21^4\}_0$	$(\!(21^4)\!)_0$
$\{22\}_{20}$	$(\!(22)\!)_{20}$	$\{221\}_{20}$	$(\!(221)\!)_{20}$
$\{2211\}_6$	$(\!(2211)\!)_6$	$\{22111\}_0$	$(\!(22111)\!)_0$
$\{3\}_{20}$	$(\!(3)\!)_{19} + (\!(0)\!)_1$	$\{31\}_{45}$	$(\!(31)\!)_{41} + (\!(1)\!)_4$
$\{311\}_{36}$	$(\!(311)\!)_{30} + (\!(11)\!)_6$	$\{3111\}_{10}$	$(\!(3111)\!)_6 + (\!(111)\!)_4$
$\{31^4\}_0$	$(\!(31^4)\!)_{-1} + (\!(1^4)\!)_1$	$\{32\}_{60}$	$(\!(32)\!)_{50} + (\!(2)\!)_{10}$
$\{321\}_{64}$	$(\!(321)\!)_{44} + (\!(21)\!)_{20}$	$\{3211\}_{20}$	$(\!(3211)\!)_5 + (\!(211)\!)_{15}$
$\{32111\}_0$	$(\!(32111)\!)_{-4} + (\!(2111)\!)_4$	$\{33\}_{50}$	$(\!(33)\!)_{31} + (\!(3)\!)_{19}$
$\{331\}_{60}$	$(\!(331)\!)_{19} + (\!(31)\!)_{41}$	$\{3311\}_{20}$	$(\!(3311)\!)_{-10} + (\!(311)\!)_{30}$
$\{33111\}_0$	$(\!(33111)\!)_{-6} + (\!(3111)\!)_6$	$\{331^4\}_0$	$(\!(331^4)\!)_1 + (\!(31^4)\!)_{-1}$
$\{4\}_{35}$	$(\!(4)\!)_{31} + (\!(1)\!)_4$	$\{41\}_{84}$	$(\!(41)\!)_{68} + (\!(2)\!)_{10} + (\!(11)\!)_6$
$\{411\}_{70}$	$(\!(411)\!)_{46} + (\!(21)\!)_{20} + (\!(111)\!)_4$	$\{4111\}_{20}$	$(\!(4111)\!)_4 + (\!(211)\!)_{15} + (\!(1^4)\!)_1$
$\{41^4\}_0$	$(\!(41^4)\!)_{-4} + (\!(2111)\!)_4 + (\!(1^5)\!)_0$	$\{42\}_{126}$	$(\!(42)\!)_{86} + (\!(3)\!)_{19} + (\!(21)\!)_{20} + (\!(0)\!)_1$

(C.1)

From this table one could start to derive modification rules. If we denote a fully symmetric tensor as η, then $(\!(\lambda_1, \lambda_2, \ldots, \lambda_n)\!)$, some $\lambda_i \geqslant 3$ can be contracted with η. Hence, the branching $\{3\}_{20} \downarrow (\!(3)\!)_{19} + \eta(\!(0)\!)_1$ extracts a triply contracted 'trace' with respect to η. However, there is still some freedom, since we find 20 rank-three fully symmetric tensors in dimension 4. A more detailed investigation of these affairs is postponed for another publication, since without a theoretical device this task is tied to tedious case-by-case studies.

Some examples of product formulae for $\mathsf{H}_3(4)$ *characters:*

$\cdot$	$(\!(1)\!)_4$
$(\!(1)\!)_4$	$(\!(2)\!)_{10} + (\!(11)\!)_6$
$(\!(2)\!)_{10}$	$(\!(3)\!)_{19} + (\!(21)\!)_{20} + (\!(0)\!)_1$
$(\!(11)\!)_6$	$(\!(21)\!)_{20} + (\!(111)\!)_4$
$(\!(3)\!)_{19}$	$(\!(4)\!)_{31} + (\!(31)\!)_{41} + (\!(1)\!)_4$
$(\!(21)\!)_{20}$	$(\!(31)\!)_{41} + (\!(22)\!)_{20} + (\!(211)\!)_{15} + (\!(1)\!)_4$
$(\!(111)\!)_4$	$(\!(21^2)\!)_{15} + (\!(1^4)\!)_1$

$\cdot$	$(\!(2)\!)_{10}$
$(\!(2)\!)_{10}$	$(\!(4)\!)_{31} + (\!(31)\!)_{41} + (\!(22)\!)_{20} + 2(\!(1)\!)_4$
$(\!(11)\!)_6$	$(\!(31)\!)_{41} + (\!(211)\!)_{15} + (\!(1)\!)_4$
$(\!(3)\!)_{19}$	$(\!(5)\!)_{46} + (\!(41)\!)_{68} + (\!(32)\!)_{50} + 2(\!(2)\!)_{10} + (\!(11)\!)_6$
$(\!(21)\!)_{20}$	$(\!(41)\!)_{68} + (\!(32)\!)_{50} + (\!(311)\!)_{30} + (\!(221)\!)_{20} + 2(\!(2)\!)_{10} + 2(\!(11)\!)_6$
$(\!(111)\!)_4$	$(\!(311)\!)_{30} + (\!(2111)\!)_4 + (\!(11)\!)_6$

$\cdot$	$(\!(11)\!)_6$
$(\!(11)\!)_6$	$(\!(22)\!)_{20} + (\!(211)\!)_{15} + (\!(1^4)\!)_1$
$(\!(3)\!)_{19}$	$(\!(41)\!)_{68} + (\!(311)\!)_{30} + (\!(2)\!)_{10} + (\!(11)\!)_6$
$(\!(21)\!)_{20}$	$(\!(32)\!)_{50} + (\!(311)\!)_{30} + (\!(221)\!)_{20} + (\!(2111)\!)_4 + (\!(2)\!)_{10} + (\!(11)\!)_6$
$(\!(111)\!)_4$	$(\!(221)\!)_{20} + (\!(2111)\!)_4$

$\cdot$	$(\!(3)\!)_{19}$
$(\!(3)\!)_{19}$	$(\!(6)\!)_{64} + (\!(51)\!)_{101} + (\!(42)\!)_{86} + (\!(33)\!)_{31} + 2(\!(3)\!)_{19} + 2(\!(21)\!)_{20} + (\!(0)\!)_1$
$(\!(21)\!)_{20}$	$(\!(51)\!)_{101} + (\!(42)\!)_{86} + (\!(411)\!)_{46} + (\!(321)\!)_{44} + 2(\!(3)\!)_{19} + 3(\!(21)\!)_{20} + (\!(111)\!)_4 + (\!(0)\!)_1$
$(\!(111)\!)_4$	$(\!(411)\!)_{46} + (\!(3111)\!)_6 + (\!(21)\!)_{20} + (\!(111)\!)_4$

$\cdot$	$(\!(21)\!)_{20}$
$(\!(21)\!)_{20}$	$(\!(42)\!)_{86} + (\!(411)\!)_{46} + (\!(33)\!)_{31} + 2(\!(321)\!)_{44} + (\!(3111)\!)_6 + (\!(222)\!)_{10} + (\!(2211)\!)_6$
	$+ 2(\!(3)\!)_{19} + 4(\!(21)\!)_{20} + 2(\!(111)\!)_4 + (\!(0)\!)_1$
$(\!(111)\!)_4$	$(\!(321)\!)_{44} + (\!(3111)\!)_6 + (\!(2211)\!)_6 + (\!(21)\!)_{20} + (\!(111)\!)_4$

$\cdot$	$((111))_4$
$((111))_4$	$((222))_{10} + ((2211))_6$

C.2. $\mathsf{H}_{21}(4)$ *formal characters*

Dimensions of $\mathsf{GL}(4) \downarrow \mathsf{H}_{21}(4)$ *irreps:*

$\{\lambda\}_{\dim}$	$((\lambda/M_{21}))_{\dim}$
$\{0\}_1$	$((0))_1$
$\{1\}_4$	$((1))_4$
$\{11\}_6$	$((11))_6$
$\{111\}_4$	$((111))_4$
$\{1^4\}_1$	$((1^4))_1$
$\{2\}_{10}$	$((2))_{10}$
$\{21\}_{20}$	$((21))_{19} + ((0))_1$
$\{211\}_{15}$	$((211))_{11} + ((1))_4$
$\{2111\}_4$	$((2111))_{-2} + ((11))_6$
$\{21^4\}_0$	$((21^4))_{-4} + ((111))_4$
$\{21^5\}_0$	$((21^5))_{-1} + ((1^4))_1$
$\{22\}_{20}$	$((22))_{16} + ((1))_4$
$\{221\}_{20}$	$((221))_4 + ((2))_{10} + ((11))_6$
$\{2211\}_6$	$((2211))_{-17} + ((21))_{19} + ((111))_4$
$\{22111\}_0$	$((22111))_{-12} + ((211))_{11} + ((1^4))_1$
$\{221^4\}_0$	$((221^4))_2 + ((2111))_{-2}$
$\{2^21^5\}_0$	$((2^21^5))_4 + ((21^4))_{-4}$
$\{2^21^6\}_0$	$((2^21^6))_1 + ((21^5))_{-1}$
$\{3\}_{20}$	$((3))_{20}$
$\{31\}_{45}$	$((31))_{41} + ((1))_4$
$\{311\}_{36}$	$((311))_{20} + ((2))_{10} + ((11))_6$
$\{31^3\}_{10}$	$((31^3))_{-14} + ((21))_{19} + ((111))_4 + ((0))_1$
$\{31^4\}_0$	$((31^4))_{-16} + ((211))_{11} + ((1^4))_1 + ((1))_4$
$\{31^5\}_0$	$((31^5))_{-4} + ((21^3))_{-2} + ((11))_6$
$\{32\}_{60}$	$((32))_{44} + ((2))_{10} + ((11))_6$
$\{321\}_{64}$	$((321))_1 + ((3))_{20} + 2((21))_{19} + ((111))_4 + ((0))_1$
$\{3211\}_{20}$	$((3211))_{-68} + ((31))_{41} + ((22))_{16} + 2((211))_{11} + ((1^4))_1 + 2((1))_4$
$\{321^3\}_0$	$((321^3))_{-42} + ((311))_{20} + ((221))_4 + 2((21^3))_{-2} + ((2))_{10} + 2((11))_6$
$\{321^4\}_0$	$((321^4))_{12} + ((31^3))_{-14} + ((2211))_{-17} + 2((21^4))_{-4} + ((21))_{19} + 2((1^3))_4$
$\{321^5\}_0$	$((321^5))_{17} + ((31^4))_{-16} + ((221^3))_{-12} + 2((21^5))_{-1} + ((211))_{11} + 2((1^4))_1$
$\{321^6\}_0$	$((321^6))_4 + ((31^5))_{-4} + ((2^21^4))_2 + ((21^3))_{-2}$
$\{33\}_{50}$	$((33))_{31} + ((21))_{19}$

(C.2)

Some examples of product formulae for $\mathsf{H}_{21}(4)$ *characters:*

$\cdot$	$((1))_4$
$((1))_4$	$((2))_{10} + ((11))_6$
$((2))_{10}$	$((3))_{20} + ((21))_{19} + ((0))_1$
$((11))_6$	$((21))_{19} + ((111))_4 + ((0))_1$
$((3))_{20}$	$((4))_{35} + ((31))_{41} + ((1))_4$
$((21))_{19}$	$((31))_{41} + ((22))_{16} + ((211))_{11} + 2((1))_4$
$((111))_4$	$((211))_{11} + ((1^4))_1 + ((1))_4$

$\cdot$	$((2))_{10}$
$((2))_{10}$	$((4))_{35} + ((31))_{41} + ((22))_{16} + 2((1))_4$
$((11))_6$	$((31))_{41} + ((211))_{11} + 2((1))_4$
$((3))_{20}$	$((5))_{56} + ((41))_{74} + ((32))_{44} + 2((2))_{10} + ((11))_6$
$((21))_{19}$	$((41))_{74} + ((32))_{44} + ((311))_{20} + ((221))_4 + 3((2))_{10} + 3((11))_6$
$((111))_4$	$((311))_{20} + ((2111))_{-2} + ((2))_{10} + 2((11))_6$

$\cdot$	$((11))_6$
$((11))_6$	$((22))_{16} + ((211))_{11} + ((1^4))_1 + 2((1))_4$
$((3))_{20}$	$((41))_{74} + ((311))_{20} + 2((2))_{10} + ((11))_6$
$((21))_{19}$	$((32))_{44} + ((311))_{20} + ((221))_4 + ((2111))_{-2} + 3((2))_{10} + 3((11))_6$
$((111))_4$	$((221))_4 + ((2111))_{-2} + ((2))_{10} + 2((11))_6$

$\cdot$	$((3))_{20}$
$((3))_{20}$	$((6))_{84} + ((51))_{120} + ((42))_{86} + ((33))_{31} + 2((3))_{20} + 2((21))_{19} + ((0))_1$
$((21))_{19}$	$((51))_{120} + ((42))_{86} + ((411))_{31} + ((321))_1 + 3((3))_{20} + 4((21))_{19} + ((111))_4 + 2((0))_1$
$((111))_4$	$((411))_{31} + ((3111))_{-14} + ((3))_{20} + 2((21))_{19} + ((111))_4 + ((0))_1$

$\cdot$	$((21))_{19}$
$((21))_{19}$	$((42))_{86} + ((411))_{31} + ((33))_{31} + 2((321))_1 + ((3111))_{-14} + ((222))_{-10} + ((2211))_{-17}$
	$+4((3))_{20} + 8((21))_{19} + 4((111))_4 + 4((0))_1$
$((111))_4$	$((321))_1 + ((3111))_{-14} + ((3))_{20} + ((2211))_{-17} + ((21^4))_{-4}$
	$+4((21))_{19} + 3((1^3))_4 + 2((0))_0$

$\cdot$	$(\!(111)\!)_4$
$(\!(111)\!)_4$	$(\!(222)\!)_{-10} + (\!(2211)\!)_{-17} + (\!(21^4)\!)_{-4} + 2(\!(21)\!)_{19} + 2(\!(111)\!)_4 + (\!(0)\!)_1$

C.3. $\mathsf{H}_{1^3}(4)$ *formal characters*

Dimensions of $\mathsf{GL}(4) \downarrow \mathsf{H}_{1^3}(4)$ *irreps:*

$\{\lambda\}_{\rm dim}$	$(\!(\lambda/M_{21})\!)_{\rm dim}$
$\{0\}_1$	$(\!(0)\!)_1$
$\{1\}_4$	$(\!(1)\!)_4$
$\{11\}_6$	$(\!(11)\!)_6$
$\{111\}_4$	$(\!(111)\!)_3 + (\!(0)\!)_1$
$\{1^4\}_1$	$(\!(1^4)\!)_{-3} + (\!(1)\!)_4$
$\{1^5\}_0$	$(\!(1^5)\!)_{-6} + (\!(11)\!)_6$
$\{1^6\}_0$	$(\!(1^6)\!)_{-3} + (\!(1^3)\!)_3$
$\{1^7\}_0$	$(\!(1^7)\!)_3 + (\!(1^4)\!)_{-3}$
$\{2\}_{10}$	$(\!(2)\!)_{10}$
$\{21\}_{20}$	$(\!(21)\!)_{20}$
$\{211\}_{15}$	$(\!(211)\!)_{11} + (\!(1)\!)_4$
$\{2111\}_4$	$(\!(2111)\!)_{-12} + (\!(2)\!)_{10} + (\!(11)\!)_6$
$\{21^4\}_0$	$(\!(21^4)\!)_{-24} + (\!(21)\!)_{20} + (\!(1^3)\!)_3 + (\!(0)\!)_1$
$\{21^5\}_0$	$(\!(21^5)\!)_{-12} + (\!(211)\!)_{11} + (\!(1^4)\!)_{-3} + (\!(1)\!)_4$
$\{21^6\}_0$	$(\!(21^6)\!)_{12} + (\!(21^3)\!)_{-12} + (\!(1^5)\!)_{-6} + (\!(11)\!)_6$
$\{22\}_{20}$	$(\!(22)\!)_{20}$
$\{221\}_{20}$	$(\!(221)\!)_{14} + (\!(11)\!)_6$
$\{2211\}_6$	$(\!(2211)\!)_{-17} + (\!(21)\!)_{20} + (\!(1^3)\!)_3$
$\{221^3\}_0$	$(\!(221^3)\!)_{-32} + (\!(22)\!)_{20} + (\!(211)\!)_{11} + (\!(1^4)\!)_{-3} + (\!(1)\!)_4$
$\{221^4\}_0$	$(\!(221^4)\!)_{-12} + (\!(221)\!)_{14} + (\!(21^3)\!)_{-12} + (\!(2)\!)_{10} + (\!(1^5)\!)_{-6} + (\!(11)\!)_6$
$\{221^5\}_0$	$(\!(221^5)\!)_{21} + (\!(2211)\!)_{-17} + (\!(21^4)\!)_{-24} + (\!(21)\!)_{20} + (\!(1^6)\!)_{-3} + (\!(1^3)\!)_3$
$\{221^6\}_0$	$(\!(221^6)\!)_{33} + (\!(221^3)\!)_{-32} + (\!(21^5)\!)_{-12} + (\!(211)\!)_{11} + (\!(1^7)\!)_3 + (\!(1^4)\!)_{-3}$
$\{2^3\}_{10}$	$(\!(2^3)\!)_6 + (\!(1^3)\!)_3 + (\!(0)\!)_1$
$\{2^31\}_4$	$(\!(2^31)\!)_{-8} + (\!(211)\!)_{11} + (\!(1^4)\!)_{-3} + (\!(1)\!)_4$
$\{2^31^2\}_0$	$(\!(2^31^2)\!)_{-8} + (\!(221)\!)_{14} + (\!(21^3)\!)_{-12} + (\!(1^5)\!)_{-6} + 2(\!(11)\!)_6$
$\{2^31^3\}_0$	$(\!(2^31^3)\!)_{11} + (\!(2^3)\!)_6 + (\!(2211)\!)_{-17} + (\!(21^4)\!)_{-24} + (\!(21)\!)_{20}$
	$+ (\!(1^6)\!)_{-3} + 2(\!(1^3)\!)_3 + (\!(0)\!)_1$
$\{2^4\}_1$	$(\!(2^4)\!)_3 + (\!(21^3)\!)_{-12} + (\!(2)\!)_{10}$

(C.3)

Some examples of product formulae for $\mathsf{H}_{1^3}(4)$ *characters:*

$\cdot$	$((1))_4$
$((1))_4$	$((2))_{10} + ((11))_6$
$((2))_{10}$	$((3))_{20} + ((21))_{20}$
$((11))_6$	$((21))_{20} + ((111))_3 + ((1))_1$
$((3))_{20}$	$((4))_{35} + ((31))_{45}$
$((21))_{20}$	$((31))_{45} + ((22))_{20} + ((211))_{11} + ((1))_4$
$((111))_3$	$((21^2))_{11} + ((1^4))_{-3} + ((1))_4$

$\cdot$	$((2))_{10}$
$((2))_{10}$	$((4))_{35} + ((31))_{45} + ((22))_{30}$
$((11))_6$	$((31))_{45} + ((211))_{11} + ((1))_4$
$((3))_{20}$	$((5))_{56} + ((41))_{84} + ((32))_{60}$
$((21))_{20}$	$((41))_{84} + ((32))_{60} + ((311))_{26} + ((221))_{14} + ((2))_{10} + ((11))_6$
$((111))_3$	$((311))_{26} + ((2111))_{-12} + ((2))_{10} + ((11))_6$

$\cdot$	$((11))_6$
$((11))_6$	$((22))_{20} + ((211))_{11} + ((1^4))_{-3} + 2((1))_4$
$((3))_{20}$	$((41))_{84} + ((311))_{26} + ((2))_{10}$
$((21))_{20}$	$((32))_{60} + ((311))_{26} + ((221))_{14} + ((2111))_{-12} + 2((2))_{10} + 2((11))_6$
$((111))_3$	$((221))_{14} + ((2111))_{-12} + ((1^5))_{-5} + ((2))_{10} + 2((11))_6$

$\cdot$	$((3))_{20}$
$((3))_{20}$	$((6))_{84} + ((51))_{140} + ((42))_{126} + ((33))_{50}$
$((21))_{20}$	$((51))_{140} + ((42))_{126} + ((411))_{50} + ((321))_{44} + ((3))_{20} + ((21))_{20}$
$((111))_3$	$((411))_{50} + ((3111))_{-30} + ((3))_{20} + ((21))_{20}$

$\cdot$	$((21))_{20}$
$((21))_{20}$	$((42))_{126} + ((411))_{50} + ((33))_{50} + 2((321))_{44} + ((3111))_{-30} + 2((3))_{20} + ((222))_6 +$
	$+ ((2211))_{-17} + 4((21))_{20} + 2((111))_3 + ((0))_1$
$((111))_3$	$((321))_{44} + ((3111))_{-30} + ((3))_{20} + ((2211))_{-17} + ((2211))_{-20}$
	$+ 3((21))_{20} + 2((111))_3 + ((0))_1$

$\cdot$	$((111))_3$
$((111))_3$	$((222))_6 + ((2211))_{-17} + ((21^4))_{-20} + 2((21))_{20} + ((1^6))_{-3} + 2((1^3))_3 + ((0))_1$

References

[1] Abłamowicz R and Fauser B 2005 Clifford and Graßmann Hopf algebras via the BIGEBRA package for Maple *Comput. Phys. Commun.* **170** 115–30

[2] Barcelo H and Ram A 1999 *Combinatorial Representation Theory (Math. Sci. Res. Inst. Publ. 38)* (Cambridge, Berkeley, CA: Cambridge University Press) pp 23–90

[3] Black G R E, King R C and Wybourne B G 1983 Kronecker products for compact semisimple Lie groups *J. Phys. A: Math. Gen.* **16** 1555–89

[4] Brouder C, Fauser B, Frabetti A and Oeckl R 2004 Quantum field theory and Hopf algebra cohomology [formerly 'Let's twist again'] *J. Phys. A: Math. Gen.* **37** 5895–927 (*Preprint* hep-th/0311253)

[5] Brouder C, Frabetti A and Krattenthaler C 2004 Non-commutative Hopf algebra of formal diffeomorphisms p 33 *Preprint* math.QA/0406117

[6] Brouder C and Schmitt W 2002 Quantum groups and quantum field theory III. Renormalization pp 1–18 *Preprint* hep-th/0210097

[7] Carvalho M J and D'Agostino S 2001 Plethysm of Schur functions and the shell model *J. Phys. A: Math. Gen.* 1375–92

[8] Carvalho M J and D'Agostino S 2001 A MAPLE program for calculations with Schur functions *Comput. Phys. Commun.* **141** 282–95

[9] Dubois-Violette M and Henneaux M 2002 Tensor fields of mixed Young symmetry type and n-complexes *Commun. Math. Phys.* **226** 393–418

[10] Fauser B 2002 A treatise on quantum Clifford algebras. Konstanz. Habilitationsschrift *Preprint* math.QA/0202059

[11] Fauser B and Jarvis P D 2004 A Hopf laboratory for symmetric functions *J. Phys. A: Math. Gen:* **37** 1633–63 (*Preprint* math-ph/0308043)

[12] Fauser B and Jarvis P D 2006 The Hopf algebra of plethysm *in preparation*

[13] Geissinger L 1977 *Hopf Algebras of Symmetric Functions and Class Functions (Springer Lecture Notes)* vol 579 pp 168–81

[14] Hasse R W and Johnson N F 1993 Classification of N-electron states in a quantum dot *Phys. Rev.* B **48** 1583–1594

[15] Kassel C 1995 *Quantum Groups* (New York: Springer)

[16] King R C 1971 Modification rules and products of irreducible representations of the unitary, orthogonal, and symplectic groups *J. Phys. A: Math. Gen.* **12** 1588–98

[17] King R C 1975 Branching rules for classical Lie groups using tensor and spinor methods *J. Phys. A: Math. Gen.* **8** 429–49

[18] King R C, Dehuai L and Wybourne B G 1981 Symmetrized powers of rotation group representations *J. Phys. A: Math. Gen.* **14** 2509–38

[19] King R C and Wybourne B G 2002 Analogies between finite-dimensional irreducible representations of $SO(2n)$ and infinite-dimensional irreducible representations of $Sp(2n, \mathbb{R})$: I. Characters and products *J. Phys. A: Math. Gen.* **41** 5002–19

[20] King R C and Wybourne B G 2002 Analogies between finite-dimensional irreducible representations of $SO(2n)$ and infinite-dimensional irreducible representations of $Sp(2n, \mathbb{R})$: II. Plethysms *J. Phys. A: Math. Gen.* **41** 5656–90

[21] King R C and Wybourne B G 1985 Holomorphic discrete series and harmonic series unitary irreducible representations of non-compact Lie groups: $Sp(2n, R), U(p, q)$ and $SO^*(2n)$ *J. Phys. A: Math. Gen.* **18** 3113–39

[22] Knutson D 1973 *λ-Rings and the Representation Theory of the Symmetric Group* (Berlin: Springer) Lecture Notes in Mathematics 308

[23] Kuperberg G 1991 Involutory Hopf algebras and 3-manifold invariants *Int. J. Math.* **2** 41–66

[24] Kuperberg G 1996 Noninvolutory Hopf algebras and 3-manifold invariants *Duke Math. J.* **84** 83–129

[25] Littlewood D E 1940 *The Theory of Group Characters* (Oxford: Oxford University Press)

[26] Littlewood D E 1950 *University Algebra* (London: Heinemann)

[27] Littlewood D E 1958 *The Theory of Group Characters* 2nd edn (Oxford: Oxford University Press)

[28] Littlewood D E 1958 Products and plethysms of characters with orthogonal, symplectic and symmetric groups *Can. J. Math.* **10** 17–32

[29] Littlewood D E 1958 The inner plethysm of S-functions *Can. J. Math.* **10** 1–16

[30] Lyubashenko V 1995 Modular transformations for tensor categories *J. Pure Appl. Algebra* **98** 279–327

[31] Lyubashenko V 1995 Tangles and Hopf algebras in braided categories *J. Pure Appl. Algebra* **98** 245–78

[32] Macdonald I G 1979 *Symmetric Functions and Hall Polynomials* (Oxford: Clarendon) (1995 2nd edn)

[33] Majid S 1995 *Foundations of Quantum Group Theory* (Cambridge: Cambridge University Press)
[34] Scharf T and Thibon J-Y 1994 A Hopf algebra approach to inner plethysm *Adv. Math.* **104** 30–58
[35] Sweedler M E 1968 Cohomology of algebras over Hopf algebras *Trans. Am. Math. Soc.* **133** 205–39
[36] Thibon J-Y 1991 Coproduits de fonctions symétrique *C. R. Acad. Sci., Paris* I **312** 553–6
[37] Thibon J-Y 1991 Hopf algebras of symmetric functions and tensor products of symmetric group representations *Int. J. Algebra Comput.* **1** 2007–221
[38] Weyl H 1930 *The classical groups, their invariants and representations by Hermann Weyl* (Princeton, NJ: Princeton University Press) (1946 2nd edn, with supplement)
[39] Wybourne B G *et al* 2004 http://smc.vnet.net/schur.html SCHUR©, Schur Group Theory Software
[40] Yang M and Wybourne B G 1986 New *S* function series and non-compact Lie groups *J. Phys. A: Math. Gen.* **19** 3513–25
[41] Yetter D N 1990 Quantum groups and representations of monoidal categories *Math. Proc. Camb. Phil. Soc.* **108** 261–90
[42] Zelevinsky A V 1981 *Representations of Finite Classical Groups: A Hopf Algebra Approach (LNM 869)* (Berlin: Springer)
[43] Zelevinsky A V 1981 A generalization of the Littlewood–Richardson rule and the Robinson-Schensted–Knuth correspondence *J. Algebra* **69** 82–94